Some short notes on a price index of Jacek Białek

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In what follows we present a price index formula proposed recently by J. Białek (. The formula is a bit unusual and unorthodox, yet quite interesting from a theoretic point of view. Some properties of this index are astonishing and unexpected, however, as a whole the index does not seem to be useful for the practical work a statistical agency.

1. Definition of Białek's price index

Jacek Białek (University of Łodz) proposed the following price index[1]

(1) $I_B^P = \left(\frac{f_1(Q^s, Q^t)P^t}{f_2(Q^s, Q^t)P^s} \frac{f_2(Q^s, Q^t)P^t}{f_1(Q^s, Q^t)P^s}\right)^{1/2} = \sqrt{I_L^P I_U^P}$,

where $f_1(\ )$ and $f_2(\ )$ are row vectors as follows

$f_1 = f_1(Q^s, Q^t) = [\min(q_1^s, q_1^t) \quad \dots \quad \min(q_N^s, q_N^t)]$ and

$f_2 = f_2(Q^s, Q^t) = [\max(q_1^s, q_1^t) \quad \dots \quad \max(q_N^s, q_N^t)]$,

where q_i^s and q_i^t ($I = 1, \dots, N$) are elements of the vectors Q^s and Q^t respectively of quantities and P^s and P^t are $N \times 1$ column vectors[2] of base period (s) prices and current period (t) prices of the N commodities. Because $I_B^P = \left(\frac{f_1(Q^s,Q^t)P^t}{f_1(Q^s,Q^t)P^s} \frac{f_2(Q^s,Q^t)P^t}{f_2(Q^s,Q^t)P^s}\right)^{1/2}$ we can also write

(1a) $I_B^P = \left(\prod_{j=1}^m \frac{f_j(Q^s,Q^t)P^t}{f_j(Q^s,Q^t)P^s}\right)^{1/m}$ which allows for a more general formula (if $m > 2$).

Białek calls $I_L^P = I_L$ lower and $I_U^P = I_U$ upper price index.[3] It can easily be seen that such labels (i.e. " upper" and " lower") are justified. Assume (without loss of generality) all elements of Q^s are equal to the corresponding elements of Q^t except one of them, say the quantity of the i-th commodity for which applies $q_i^s < q_i^t$ (of course we could also assume $q_i^s > q_i^t$ and interchange < and > in the following) Then

(2) $f_1P^t < f_2P^t$ and $f_1P^s < f_2P^s$,

whatever the prices in s and t may be, so that the numerator of I_L (which is f_1P^t) is smaller than the numerator of I_U (f_2P^t) and the opposite applies to the denominators (f_2P^s of I_L, and f_1P^s of I_U respectively), so we may conclude

(3) $I_L < I_U$.

We may now introduce the vectors $g_s = [q_1^s \quad \dots \quad q_N^s]$ and $g_t = [q_1^t \quad \dots \quad q_N^t]$ so that we can define the price index functions of Laspeyres and Paasche

(4) $I_{La} = g_sP^t/g_sP^s$, and $I_{Pa} = g_tP^t/g_tP^s$.

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[1] Białek 2012c. We adopt Białek's notation(though quite different from ours) as far as it appears reasonable and convenient.

[2] Thus fP represents a scalar product (a real number).

[3] We simplify the notation of Białek a bit because in what follows we only deal with price indices (and not with quantity indices which then should be denoted by I^Q consequently). We also drop all arguments of index functions and write for example simply I_L instead of $I_L^P(Q^s, Q^t, P^s, P^t)$ all the time.

Assume $q_i^s < q_i^t$ while for all other N-1 commodities (j) $q_j^s = q_j^t$ holds (or alternatively assume that for all I = 1, …, N we have $q_i^s < q_i^t$). Under these conditions we have $f_1 = g_s$, and $f_2 = g_t$ so that

(5a) $I_L = \frac{g_s P^t}{g_t P^s} \leq I_{La} = \frac{g_s P^t}{g_s P^s} \leq I_U = \frac{g_t P^t}{g_s P^s}$

because $g_t P^s > g_s P^s$ ($\rightarrow I_L < I_{La}$) and $g_s P^t < g_t P^t$ ($\rightarrow I_{La} < I_U$) and

(5b) $I_L = \frac{g_s P^t}{g_t P^s} \leq I_{Pa} = \frac{g_t P^t}{g_t P^s} \leq I_U = \frac{g_t P^t}{g_s P^s}$

for the same reason[4]. From (5a) and (5b) it follows that I_{La} (and also I_{Pa}) can be expressed as geometric mean (or any other mean, e.g. arithmetic mean) of I_L and I_U.

However, it is not clear whether $I_{La} \geq I_{Pa}$ or $I_{La} \leq I_{Pa}$ is true. This depends – according to a well known theorem of L. v. Bortkiewicz[5] – on the sign of the covariance between price and quantity relatives, that is p_i^t/p_i^s and q_i^t/q_i^s respectively.

Furthermore under such conditions the famous "ideal index" of Fisher coincides with Białek's index, since

(6) $$I_F = \sqrt{I_{La} I_{Pa}} = \sqrt{\frac{g_s P^t}{g_s P^s} \frac{g_t P^t}{g_t P^s}} = \sqrt{\frac{g_s P^t}{g_t P^s} \frac{g_t P^t}{g_s P^s}} = \sqrt{I_L I_U} = I_B.$$

It will be seen, however, that under other conditions than those assumed above I_B may (in general) well differ from I_F. From a practical point of view it may not be very useful to write I_{La} or I_{Pa} as weighted mean of I_L and I_U,[6] but it is easy and quite interesting to see that the Marshall Edgeworth price index[7] defined as

$I_{ME} = (g_s+g_t)P^t/(g_s+g_t)P^s$

can be written as both, a weighted arithmetic mean of I_L and I_U

(7) $$I_{ME} = \frac{g_s P^t + g_t P^t}{g_s P^s + g_t P^s} = \frac{\sum p_i^s q_{max}}{\sum p_i^s q_{max} + \sum p_i^s q_{min}} I_L + \frac{\sum p_i^s q_{min}}{\sum p_i^s q_{max} + \sum p_i^s q_{min}} I_U,$$

using the fact that $f_1 P^s + f_2 P^s = \sum p_i^s q_{max} + \sum p_i^s q_{min} = (g_s+g_t)P^s$ (and the analogous relation holds for P^t), as well as a weighted arithmetic mean of I_{La} and I_{Pa}.

(7a) $$I_{ME} = \frac{1}{1 + I_{La}^Q} I_{La} + \frac{I_{La}^Q}{1 + I_{La}^Q} I_{Pa},$$

where I_{La}^Q denotes the quantity index of Laspeyres, and $I_{La} = I_{La}^P$ the respective price index.

Equation (7) shows that we may well relate the components of I_L and I_U and therefore also Białek's index to the sum (or unweighted average) of quantities in both periods, s *and* t, that is to aggregates like $(g_s+g_t)P^t$, or $(g_s+g_t)P^s$ but not to quantities relating to one period only, say g_t only. We therefore cannot relate Białek's formulas I_L, I_U, or I_B to the value aggregates (price-

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[4] From $g_s P^t < g_t P^t$ follows $I_L < I_{Pa}$, and from $g_t P^s > g_s P^s$ follows $I_{Pa} < I_U$.
[5] See v. d. Lippe (2007), p. 194 ff.
[6] I saw that Białek made use of formulas of I_{La} and I_{Pa} as weighted geometric means of I_L and I_U. This, however, took place only for the purpose of certain proofs.
[7] I learnt from the first draft of the 2012a paper of Białek that this index also seems to be known in Poland as index of Lexis (Wilhelm Lexis 1837 – 1914 was one of the few economists in these days in Germany whose work was to a great deal devoted to mathematics, while the main stream economist were decidedly "anti-mathematicians". Bortkiewicz (1868 – 1931) was his student in Göttingen and was awarded the doctorate there in 1893).

quantity-products of a certain period) or to the value ratio (or "value index"), which should be $V_{st} = g_tP^t/g_sP^s$ as a ratio of two scalars. While V_{st} divided by I_{Pa} (Paasche prices index) gives a Laspeyres quantity index (or V_{st}/I_{La} gives a Paasche quantity index), it does not make sense to divide by I_L or I_U or I_B. Hence Białek's price index is not eligible for serving as a deflator, to deflate for example the value in order to get a "volume" (value at constant prices).

Moreover, there does not exist a quantity index of Białek. Defined analogously to the Price index it should read as follows be the geometric mean of $I_B^Q = \left(\frac{f_1(P^s,P^t)Q^t}{f_2(P^s,P^t)Q^s}\frac{f_2(P^s,P^t)Q^t}{f_1(P^s,P^t)Q^s}\right)^{1/2}$, which definitely is not the same as V_{st}/I_B^P.

2. The lower and upper index (I_L and I_U) of Białek taken in isolation

Assume two commodities, A and B with prices and quantities as follows

	prices			quantities			price-quantity-products			
	p_s	p_t	p_t/p_s	q_s	q_t	q_t/q_s	p_sq_{min}	p_tq_{min}	p_sq_{max}	p_tq_{max}
A	5	6	1.2	2	3	1.5	10	12	15	18
B	3	4	1.33	4	5	1.25	12	16	15	20
						sum	22	28	30	38

$I_L = 28/30 = 0.933$, $I_U = 38/22 = 1{,}727$. Because all quantity relatives are uniformly > 1 we have $q_{min} = q_s$ and $q_{max} = q_t$ so that $I_{La} = 28/22 = 1.273$ and $I_{Pa} = 38/30 = 1.267$, so that $I_L < I_{Pa} < I_{La} < I_U$. Note that in this case $I_B = I_F$ (Fishers index $(I_{La}I_{Pa})^{1/2}$) = 1.2697.[8]

It is well known that under fairly general conditions I_{Pa} is the lower bound of the "economic theory index" or (true) cost of living index (COLI) and I_{La} the upper bound respectively. So $I_L < I_{Pa}$ and $I_U > I_{La}$ cannot be related to the COLI concept, that is they don't have a COLI interpretation in terms of utility maximization behaviour on a given indifference curve.

Moreover I_L and I_U are not reasonable price index formulas. I_L is smaller than the smallest price relative 0.933 < 1.2, and I_U = 1.727 exceeds the greatest price relative 1.33). Hence I_L and I_U (unlike I_{Pa} and I_{La} in the case of Fisher's index) do not possess the mean value property. It is quite obvious that neither I_L nor I_U can be written as (weighted) arithmetic mean of price relatives:

$$I_L = \frac{\sum p_t q_{min}}{\sum p_s q_{max}} = \sum \frac{p_t}{p_s}\frac{p_s q_{min}}{\sum p_s q_{max}} \text{ and } I_U = \frac{\sum p_t q_{max}}{\sum p_s q_{min}} = \sum \frac{p_t}{p_s}\frac{p_s q_{max}}{\sum p_s q_{min}}.$$

In the example above we have $\Sigma p_sq_{min} = 22 < \Sigma p_sq_{max} = 30$ so that the sum of the weights is less (more) than unity in the case of I_L (I_U). Thus both components of I_B, I_L and I_U *necessarily* fail the mean value test, because *by definition* $\Sigma p_sq_{min} < \Sigma p_sq_{max}$. They cannot be viewed as price indices, as opposed to I_{La} and I_{Pa} in the case of I_F.

Let $\Sigma p_tq_{min} = A$ and $\Sigma p_tq_{max} = A + \alpha$ and analogously $\Sigma p_sq_{min} = B$ and $\Sigma p_tq_{max} = B + \beta$. We can definitely state that, $\alpha > 0$ and $\beta > 0$, and we now can see that $I_L < I_U$ is *generally* true (which, however does not apply to the pair I_{Pa} and I_{La}), because $I_L + C = I_U$ with $C > 0$, can be written as $\frac{A}{B+\beta} + C = \frac{A+\alpha}{B}$ and after solving for C

$$C = \alpha + \frac{A}{B+\beta}\cdot\frac{\beta}{B} = \alpha + I_L\cdot\frac{\beta}{B}. \tag{8}$$

[8] Note, the difference $I_U - I_L = 0.794$ is much greater than the difference $I_{La} - I_{Pa} = 0.006$, yet the geometric mean $I_B = (I_UI_L)^{1/2} = I_F = (I_{La}I_{Pa})^{1/2}$. We can easily construct examples with $I_B \neq I_F$.

Given that both terms on the right hand side are positive (in particular $\alpha > 0$ and $\beta > 0$) we see that $C > 0$ and therefore $I_L < I_U$.

Another interesting property of index is that it is invariant upon certain changes. Consider two modifications of the original example (only assumptions concerning quantities are changed, prices remain the same in all three cases)

original example		
q_s	q_t	q_t/q_s
2	3	1.5
4	5	1.25

variant 1		
q_s	q_t	q_t/q_s
2	3	1.5
5	4	0.8

variant 2		
q_s	q_t	q_t/q_s
3	2	0.67
5	4	0.8

In the first variant the quantities in s and t of commodity B are changed. In variant 2 both quantities of A and B are interchanged. The value of the minimum and maximum quantities are not affected (the vectors f_1 of the minima, and f_2 of the maxima remain unchanged). The interesting feature of Białek's indices I_L, I_U and thus also I_B now is that they remain unchanged as well, viz. $I_L = 0.933$, $I_U = 1.727$, and $I_B = 1.2697$.

While different situations may result in the same indices I_L, I_U and I_B the indices of Laspeyres and Paasche may well be quite different.

original example	
La	Pa
1.2727	1.2667

variant 1	
La	Pa
32/25 = 1.2800	34/27 = 1.2593

variant 2	
La	Pa
1.2667	1.2727

Variant 2 is simply the reverse situation of the original example. Also I_F may undergo some changes. Variant 1 yields $I_F = 1.269587$ which is slightly less than $I_B = 1.269693$.[9]

3. Interpretation of "time reversibility" in the case of Białek's index

Białek's indices require a re-interpretation of the notion of time reversibility by which is usually meant that both, P^s and P^t on the one hand as well as Q^s and Q^t on the other hand are interchanged (in symbols $P^s \leftrightarrow P^t$ and $Q^s \leftrightarrow Q^t$). Time reversibility then requires $P_{ts} = 1/P_{st}$ (s in P_{st} denotes the base, and t current the period while in P_{ts} the base period is t which is compared to the current period s).

However, as a rule neither I_L nor I_U incorporate the complete vector Q^S and Q^t respectively, so a process of interchanging $Q^s \leftrightarrow Q^t$ does not take place. Instead both, numerator and denominator of I_L and I_U contain only *some* quantities q^s and *some* quantities q^t. And this is true for I_L and I_U no matter whether the base period is taken as s or as t.

To see what this means consider an international comparison. I_{st} may represent a comparison between s = Poland and t = Germany. The I_L and I_U index combine some Polish prices P^s with German quantities and for some other goods Polish prices with Polish quantities, depending on which of the two quintiles is greater (or smaller). What does now country reversal mean?

It is clear what changes are made with respect to prices when we switch from P_{st} to P_{ts} but it appears difficult to state (in terms of verbal interpretation) what happens with the quantities.[10]

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[9] We get the opposite result, that is $I_{Pa} = 32/25$ and $I_{La} = 34/27$ (and therefore $I_{La} < I_{Pa}$) with $Q^S = (3\ 4)$ and $Q^t = (2\ 5)$ instead of (see above variant 1) $Q^s = (2\ 5)$ and $Q^t = (3\ 4)$. Interestingly this interchanging of Q-vectors (as it is common to be studied in the framework of the time reversal test) does not affect I_L and I_U.

[10] By contrast this is of course most simple in the case of I_{La} or I_{Pa}. For example in $I_{La(st)}$ we compare P^t to P^s using quantities Q^s as "weights" while $I_{La(ts)}$ means to compare prices P^s to P^t (now they are "set 100" instead of the prices P^S) using quantities Q^s as weights. As mentioned above, to imagine what it means to take this set of weights or that set of weights may be particularly easy in the case of international instead of intertemporal comparisons.

"Time" reversal now amounts to taking $\max(q_i^s, q_i^t)$ where we had $\min(q_i^s, q_i^t)$,[11] and $\min(q_i^s, q_i^t)$ where we had $\max(q_i^s, q_i^t)$ so that (interchanging also $P^s \leftrightarrow P^t$) we get f_2P^t from f_1P^s etc. which of course implies "time reversibility" as just defined. Note that it is the fact that

$$\max(q_i^s, q_i^t) = \max(q_i^t, q_i^s) \text{ (symmetry), and if } q_i^s = \max(q_i^s, q_i^t) \text{ then } q_i^t = \min(q_i^s, q_i^t),$$

which is responsible for the result that $I_{L(ts)} = (I_{L(st)})^{-1}$ and likewise $I_{U(ts)} = (I_{U(st)})^{-1}$.

Consider a function $f_j(Q^s,Q^t)$ generating a vector f_j which is not symmetric, for example $f(q_i^s, q_i^t) = aq_i^s + bq_i^t$, so that we have $f = [aq_1^s + bq_1^t \quad \ldots \quad aq_N^s + bq_N^t]$ instead of f_1 or f_2 for the vector of quantities. We then get a generalized Marshall-Edgeworth index[12] $I_{ME(st)} = \frac{\sum_i(aq_i^s+bq_i^t)p_i^t}{\sum_i(aq_i^s+bq_i^t)p_i^s} = \frac{aI_{La(st)}^P+bV_{0t}}{a+bI_{La(st)}^Q}$ (by contrast to (7a)) with price indices I^P and quantity indices I^Q, and the value ratio defined as $V_{st} = I_{La(st)}^P I_{Pa(st)}^Q = I_{Pa(st)}^P I_{La(st)}^Q$.

Interchanging s and t (in the spirit of the time reversal test) gives $I_{ME(ts)} = \frac{\sum_i(aq_i^t+bq_i^s)p_i^s}{\sum_i(aq_i^t+bq_i^s)p_i^t} = \frac{aI_{La(st)}^Q+b}{aV_{0t}+bI_{La(st)}^P}$, thus $I_{ME(st)}I_{ME(ts)} \neq 1$ unless a = b (which is the [special] ME-index as it is usual known as ME-index and considered above. So only functions $f_j(Q^s,Q^t)$ that are invariant upon interchanging q_i^s and q_i^t will result in indices that comply with time reversibility. For example in the case of a = b we get $[q_1^s + q_1^t \quad \ldots \quad q_N^s + q_N^t]$ and the special ("usual") ME index $\frac{\sum_i(q_i^s+q_i^t)p_i^t}{\sum_i(q_i^s+q_i^t)p_i^s}$ which satisfies time reversibility. However, the general ME-formula studied above does not pass the time reversal test.

It is doubtful whether time reversibility is essential (as often stated in the Anglo-American index theory, possibly as a legacy of Irving Fisher) and worth sacrificing other useful aspects of index construction, because time reversibility rules out a number of reasonable index functions, as for example Laspeyres and Paasche, to name only two.[13]

4. A final remark concerning practicalities and Fisher's index

The above mentioned idea of taking either Polish or German quantities in a comparison of national price levels (e.g. Poland as compared to Germany) depending on which quantity is smaller or greater brings us to another interesting point concerning the Białek index: It is requisite for I_L and I_U to have numerical values of "quantifiers" in physical units. However, in practice this is often not the case. As a rule we will have difficulties to define "quantity" in the case of services. Can we properly decide which amount of a certain service, for example a health, educational, or transport service, is the smaller one, the Polish or the German? Moreover, in many cases we have expenditures and weights are expenditure shares rather than quantities. From a practical point of view the left and right hand side of the equation defining I_{La} is not the same, and index compilation makes use of the right hand side of $\frac{\sum p^t q^s}{\sum p^s q^s} = \sum \frac{p^t}{p^s} \frac{p^s q^s}{\sum p^s q^s}$.
This shows that in order to be useful for the practice of (official) price statistics, a price index should have an "average of price relatives" (or "price ratios") interpretation, which is given in the case of both components of I_F, that is I_{La} and I_{Pa}, as opposed to I_B with its components I_L

[11] Interestingly this is a kind of interchanging we also have when we compare I_L to I_U.
[12] The index (7a) introduced above is simply the special case a = b = 1/2.
[13] It is praiseworthy that Białek quoted this standpoint of mine in Białek (2012c). I know that for example Diewert and myself disagree in this point, or as Diewert wrote in a private communication: We agree that we disagree in this point.

and I_U. It is, in my view at least, a considerable disadvantage of I_F that it has neither an "average of price ratios" nor a "ratio of average prices" interpretation. Nonetheless I_F enjoys a high reputation. So this defect of I_B may not be considered serious.[14] Two other shortcomings both indices have in common (and which are notoriously treated with indulgence in the case of I_F), are problems when used as deflators,[15] and poor aggregation properties (to compose an index from sub-indices or to decompose, or "disaggregate", an aggregate index into sub-indices).

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[14] However, unlike I_F we have in the case of Białek's index I_B two components I_L and I_U which cannot be interpreted as averages of price relatives.

[15] It is well known that using I_F as deflator results in volumes which are not additive. See von der Lippe 2007, p. 362 ff.