UNIVERSITÄT DUISBURG ESSEN

Prof. Dr. Peter von der Lippe
www.von-der-lippe.org

Problems with Chain Indices (I)

Introduction, general aspects (properties, arguments pro and con)

Course delivered at the European Central Bank Frankfurt

January 2010

1. Some **fundamental distinctions**
   - **two types of indices**: direct and chain
   - **two elements** of the definition of a chain index: chain and link
   - and need for a clear terminology and **notation**

2. Some common **misunderstandings**

   (1) chain index always up to date: **most recent weights**

   (2) chain index because **chaining** gives **chainability** (= transitivity)

   chain indices are gained by *chaining* (multiplying links)
   but they are not *chainable* (they violate transitivity:
   there is "chain drift", "path dependence")

   (3) chaining (**multiplying**) is **better** and a **more general approach**

3. Increasing relevance of chaining (scanner data etc.)

Types of comparison between 0 and t

direct index approach
using data of 0 and t only

1	2	3	4

1 and 4 compared directly

chain index approach
index defined as a product of links

1	2	3	4

1 and 4 compared indirectly over 3 links
1 – 2 2 – 3 3 – 4

Each index formula exists **in both forms** : chain and direct

weighted indices e.g. Laspeyres, Paasche, Fisher

chain $\overline{P}^{L}_{0t}$ $\overline{P}^{P}_{0t}$ $\overline{P}^{F}_{0t}$ direct P^{L}_{0t} P^{P}_{0t} P^{F}_{0t}

unweighted e.g. Carli, Jevons

and **as price** index P **or quantity index** Q

1.1.1 (2) Chain indices and direct indices

direct index

Definition of an price index as a function of price and quantity vectors $P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$ does not apply to chain indices and the COLI

axioms are usually defined for this situation only

chain index

A chain price index is a function of many prices and quantities

$P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_1, \mathbf{q}_1, \ldots, \mathbf{p}_{t-1}, \mathbf{q}_{t-1}, \mathbf{p}_t, \mathbf{q}_t)$

a chain index also reflects changes in the intermediate periods 1, 2, …, t-1
it is not surprisingly "path dependent"

Terms commonly used (instead of "direct") but not pertinent:

"fixed based"[1)], "fixed weighted"[2)] or "fixed basket"[3)]

1) only a link – not the chain – has a variable base
2) weights [quantities] of direct Paasche indices are no less "fixed" than weights of chain Paasche
3) applies only to direct Laspeyres

Two elements needed to **define a chain index**

constant element: **chain**
The index is gained by multiplying links

variable element: the **link**
the link is index with the *preceding period* as base period

$$\overline{P}_{0t}^{L} = P_1^L P_2^L \ldots P_t^L$$

we therefore have Laspeyres, Paasche, Fisher etc **links**, and **their product** is a Laspeyres, Paasche, Fisher etc **chain** index

$$P_{t-1,t}^{L} = P_t^L = \frac{\sum p_t q_{t-1}}{\sum p_{t-1} q_{t-1}}$$

$$P_t^P = \frac{\sum p_t q_t}{\sum p_{t-1} q_t}$$

note

the **link** is an index (complying with certain axioms), however, the **chain** is **not**

1.1.2 (2) Definition (two elements): when chain-linking is needed or not needed

P_{01}			
P_{02}			
P_{03}			
P_{04}			

Successive elements of a direct index already **form a time series**, so there is **no need to** multiply ("chain" or "**chain-link"**) them. Ideally successive indices only differ with respect to prices in the numerator

$$P_{01}^{L} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \qquad P_{02}^{L} = \frac{\sum p_2 q_0}{\sum p_0 q_0} \qquad P_{03}^{L} = \frac{\sum p_3 q_0}{\sum p_0 q_0}$$

However, a series of **links** does **not form a time series**; each link covers only part of the interval

P_1			
	P2		
		P3	
			P4

in order to form a time series links **have to be** "chain-linked"

P_{01}			
$\overline{P}_{02} = P_1 P_2$			
$\overline{P}_{03} = P_1 P_2 P_3$			
$\overline{P}_{04} = P_1 P_2 P_3 P_4$			

It appears more reasonable to multiply links, **rather than to add** them because this makes sense in the case of a single commodity i (that is in the case of s price relative)

$$\frac{p_{i4}}{p_{i0}} = \frac{p_{i1}}{p_{i0}} \frac{p_{i2}}{p_{i1}} \frac{p_{i3}}{p_{i2}} \frac{p_{i4}}{p_{i3}}$$

A solution could be to **add** the links together: P_1+P_2, and $P_1+P_2+P_3$ etc. They have, however, no common denominator.

Analogies to relatives is the legacy of I. Fisher

1.1.2 (3) Terminology: avoid "fixed base" or "fixed weighted"

Terminology "fixed": consider a **sequence** of links/indices

Index	chain/direct index		
Laspeyres chain	$\bar{P}^L_{01} = \frac{\sum p_1 q_0}{\sum p_0 q_0}$	$\bar{P}^L_{02} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \frac{\sum p_2 q_1}{\sum p_1 q_1}$	$\bar{P}^L_{03} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \frac{\sum p_2 q_1}{\sum p_1 q_1} \frac{\sum p_3 q_2}{\sum p_2 q_2}$
Laspeyres direct	$P^L_{01} = \frac{\sum p_1 q_0}{\sum p_0 q_0}$	$P^L_{02} = \frac{\sum p_2 q_0}{\sum p_0 q_0}$	$P^L_{03} = \frac{\sum p_3 q_0}{\sum p_0 q_0}$
Paasche chain	$\bar{P}^P_{01} = \frac{\sum p_1 q_1}{\sum p_0 q_1}$	$\bar{P}^P_{02} = \frac{\sum p_1 q_1}{\sum p_0 q_1} \frac{\sum p_2 q_2}{\sum p_1 q_2}$	$\bar{P}^P_{03} = \frac{\sum p_1 q_1}{\sum p_0 q_1} \frac{\sum p_2 q_2}{\sum p_1 q_2} \frac{\sum p_3 q_3}{\sum p_2 q_3}$
Paasche direct	$P^P_{01} = \frac{\sum p_1 q_1}{\sum p_0 q_1}$	$P^P_{02} = \frac{\sum p_2 q_2}{\sum p_0 q_2}$	$P^P_{03} = \frac{\sum p_3 q_3}{\sum p_0 q_3}$

In **all** cases we have **the same base** (0) of the chain (don't mistake the link for the chain!). The weights of direct Paasche are no less variable (that is **not** fixed) than weights of chain Paasche ("fixed weights" only in P^L)

1.1.3 (1) Need for a consistent and exact notation

An example for creation of utmost confusion due to inconsistent notation:

K.-H. **Tödter**, Umstellung der Deutschen VGR …, Deutsche Bundesbank, Working Paper (series 1) 31/2005

summation over?

(1) $$Q_t = \sum p_0 q_t$$

Q_t denotes an aggregate (not an index) at ***constant*** prices of period 0

Q_{t-1} then should be (1a) $$Q_{t-1} = \sum p_0 q_{t-1}$$

(2) $$P_t = \frac{\sum p_t q_t}{\sum p_0 q_t} = \frac{N_t}{Q_t}$$

Pt is the (***direct***) Paasche ***index***

P_{t-1} then should be (2a) $$P_{t-1} = \frac{\sum p_{t-1} q_{t-1}}{\sum p_0 q_{t-1}}$$

Volumes at prices of the ***preceding*** period* are

(3) $$Q_t = Q_{t-1} \frac{\sum p_{t-1} q_t}{\sum p_{t-1} q_{t-1}}$$

if this were Q_t as defined in (1) Q_{t-1} should be $$Q_{t-1} = \frac{\sum p_{t-1} q_{t-1} \sum p_0 q_t}{\sum p_{t-1} q_t}$$

quite different from (2a)!

The implicit deflator in the new chain based deflation is said to be

(4) $$P_t = P_{t-1} \frac{\sum p_{t-1} q_t}{\sum p_0 q_{t-1}}$$

by contrast to (2) this should be a *chain* index (same symbol as (2) where P is a *direct* index!!)

$$\overline{P}_{0t} = \frac{\sum p_1 q_1}{\sum p_0 q_1} \cdots \frac{\sum p_t q_t}{\sum p_{t-1} q_t}$$

obviously (4) is totally inconsistent with (2)

* Vorjahrespreisbasis mit Verkettung … Volumenaggregat durch Fortschreibung, … jeweils mit Preisen der Vorperiode bewertet

1.1.3 (2) Inconsistent notation and misunderstandings

using (2a) on the right hand side of (4) we get $P_t = \frac{\sum p_{t-1}q_{t-1} \sum p_0 q_t}{\sum p_0 q_{t-1} \sum p_t q_t}$ which does not fit to (2)

(3) $Q_t = Q_{t-1} \frac{\sum p_{t-1}q_t}{\sum p_{t-1}q_{t-1}}$ and Q_t as stated above (1) implies $Q_{t-1} = \frac{\sum p_{t-1}q_{t-1} \sum p_0 q_t}{\sum p_{t-1} q_t}$ which contradicts (1a)

(3') $Q_t = Q_{t-1} \sum v_{t-1} \frac{q_t}{q_{t-1}} = \sum \frac{p_{t-1}}{P_t} \cdot q_t$ for the second part of the equation to be correct, and Q_{t-1} as in (1a), and hence Q_t according to eq (1) P_t should be $P_t = \sum p_{t-1} q_{t-1} = N_t$

this implies $v_{t-1} = q_{t-1} p_{t-1} / \sum q_{t-1} p_{t-1}$

As a consequence: misunderstandings as for example

Tödter: **volumes** from chain indices are **additive** (can be aggregated stepwise)*

however, this is simply wrong and applies only to **links** of a chain index, not to chains, and to volumes derived from them. Therefore

It is of utmost importance to make (with a consistent notation) a distinction between

1. **aggregates** (monetary terms) and **indices**
2. **direct** indices and **chain** indices
3. a **link** (factor) for period t and the **chain** (product) for the interval 0,t

* Auch mit Kettenindizes kann die Aggregation stufenweise erfolgen … Es führt also zu demselben Ergebnis, ob ein Volumenaggregat aus mehreren Komponenten direkt berechnet wird, oder ob zuerst Teilaggregate gebildet und diese anschließend aggregiert werden. ***Please remember this wrong statement in slide 3.4.1 (2)***

1.2 Some common misunderstandings

This section deals with three wide-spread statements (very common among "chainers")

1. A chain index is always up-to-date in that it makes use of the **most recent** (most "representative", or "relevant") **weights**
2. It makes **consistent comparisons over** (long) **time by chaining** (or chain-linking, that is multiplying links to form a chain)
3. Chain indices are in a way a **more general approach than direct** (binary, comparing only two periods) indices
   a) in the links (factors) an up-date is made not only with respect to prices but also with respect to quantities
   b) the difference between direct and chain indices is basically only a difference regarding the frequency of updating of weights

The third statement will bring us to the "multiplication mystery" (argument A3 in favour of chain indices → **2.2.1**)

1.2.1 (1) **First misunderstanding**: more up-to-date weights, base

1. A chain-index always makes use of **the most recent weights**;*
2. There is **no** problem of **choosing the** correct (appropriate) **base** period, because the base period is always the previous period*

types of weights

- **single** weights (relating to one period only and fixed)
  - choice of base period may be a problem
- **multiple** weights (relating to more than one period)
  - **average** weights, mostly relating to **two** periods, 0 and t
  - **cumulative** weights, relating to **all** preceding periods

* Nr. 1 is not correct because there is no single weight. Nr. 2 is incorrect because "base = t-1" applies to the links not to the chain

A chain-index is affected by all previous weights

A chain index is a function of all vectors

p_0, q_0, $\mathbf{p}_1$, $\mathbf{q}_1$, $\mathbf{p}_2$, $\mathbf{q}_2$,...., $\mathbf{p}_{t-1}$, $\mathbf{q}_{t-1}$, p_t, q_t,

1.2.1 (2) More Up-dating of weights may have quite different meanings

See Eiglsperger/Schackis for more details (more about the obsession with more frequent up-dates sec. 4 in part II)

update of weights

price update of weights only (to December of y-1 is obligatory for the HICP)

full update of **quantities and prices** (expenditure weights)

some weights only in an ad hoc manner (e.g. some "critical" weights only)

complete revision of **all weights** (the structure of expenditures)

in a **rough** breakdown on the basis of **annual** NA data (applies to the majority of MS of the EU)

detailed structural information based on less frequent Household Expenditure Surveys (HES) in some MS (more in **4.1.1**)

Hence in practice there may not be a clear borderline between chain and direct indices

1.2.1 (3) Acceleration of the updating of chain-index weights

Much of the enthusiasm about chain indices boils down to an obsession with most recent (ideally simultaneous) weights. A question never answered

what is the difference between **"lag-zero" links**

$$\lim_{\Delta\to 0}\frac{\sum_i p_{it}q_{it-\Delta}}{\sum_i p_{i,t-1}q_{i,t-1-\Delta}}$$

tending to the links

$$\frac{\sum_i p_{it}q_{it}}{\sum_i p_{i,t-1}q_{i,t-1}}$$

which are when chain-linked resulting in the value index

$$V_{0t}=\frac{\sum p_1q_1}{\sum p_0q_0}\frac{\sum p_2q_2}{\sum p_1q_1}\cdots\frac{\sum p_tq_t}{\sum p_{t-1}q_{t-1}}=\frac{\sum p_tq_t}{\sum p_0q_0}$$

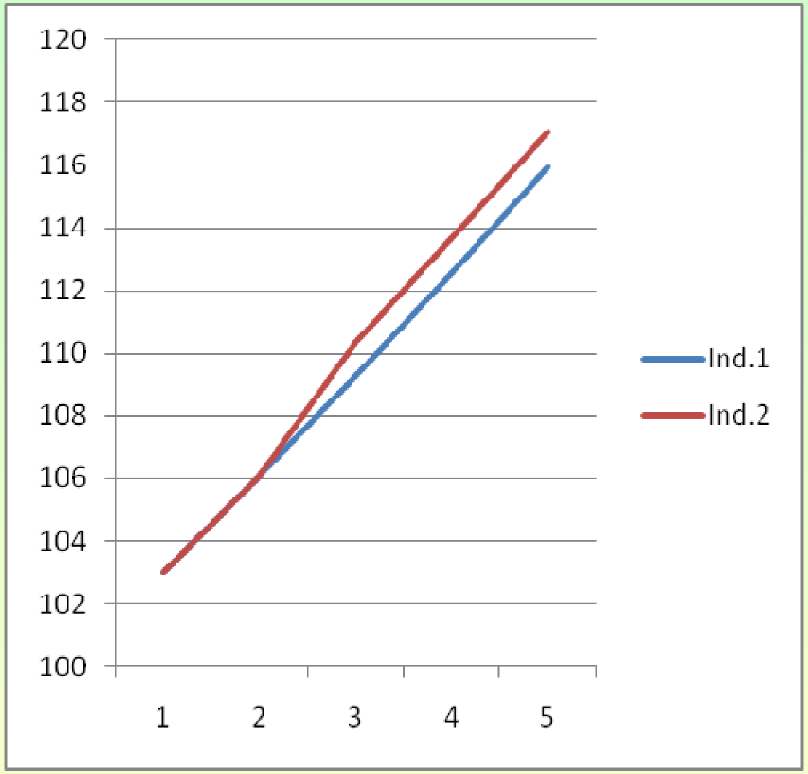
$$\frac{\sum p_1q_{1-\Delta}}{\sum p_0q_0}\frac{\sum p_2q_{2-\Delta}}{\sum p_1q_{1-\Delta}}\cdots\frac{\sum p_tq_{t-\Delta}}{\sum p_{t-1}q_{t-1-\Delta}}=\frac{\sum p_tq_{t-\Delta}}{\sum p_0q_0}$$

The value index is transitive (chainable) $V_{0s}V_{st}=V_{0t}$ but should be different from a price index P_{0t} or quantity index Q_{0t} respectively. $V_{0t}=P_{0t}Q_{0t}$ (product test), hence $V_{0t}\to P_{0t}$. Is it reasonable to have P_{0t} coming as close as possible to V_{0t}? If one strives at $\Delta\to 0$ the chained price index eventually coincides with the (always most up to date) value index.

1.2.1 (4) Demonstration of the cumulative nature of chain-index weights

The **relevance of** an as speedy as possible **update** of weights seems to be a bit **exaggerated**.

According to the German National CPI the difference between annual inflation rates for 2006 and 2007 was only about 0.1 percentage points depending on whether weights of the year 2000 or of the year 2005 were used.



However, due to multiplying links, false weights can have a **cumulative** (lasting) **effect**

Assume correct *constant* change by 3%, and a biased rate (4%) in t= 3 in index 2

Finally: Statistical institutes have to strike a balance

- SUFFICIENTLY UP-TO-DATE TO ACCOUNT FOR STRUCTURAL CHANGE
- ACCURATE; RELIABLE, NOT TOO EXPENSIVE

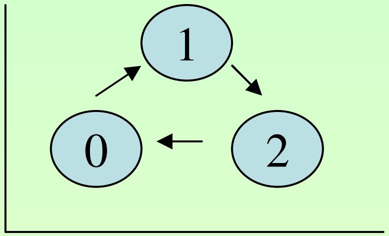
Moreover: frequency of up-date need not be the same for all groups of goods

1.2.2 (1) **Second misunderstanding**: Chaining and chainability

What is "**chainability**" (Verkettbarkeit) or "**transitivity**"?

for a *direct* index should hold → in particular "**circular test**" $P_{03} = P_{00} = 1$ (or multi-period identity) if 3 = 0

(1) $P_{03} = P_{01}P_{12}P_{23}$



as chain indices are products of links

$$\overline{P}_{0t} = \prod_{\tau=1}^{t} P_{\tau-1,\tau t} = P_{01}P_{12}\ldots P_{t-1.t}$$ and $$\overline{P}_{0t} = \overline{P}_{0k}\overline{P}_{kt}$$

Some authors therefore conclude: chain indices pass the chain-test (***chainability***, circular test) "***by construction***". However:

- eq. (1) requires $\overline{P}_{0t} = P_{0t}$ and $\overline{P}_{00} = 1$ — **no** multi-period **identity** and **drift** (away from direct index)
- and should hold ***for any partitioning*** of the interval (0, t) [idea of "intercalation" Westergaard] $P_{06} = P_{03}P_{36} = P_{02}P_{23}P_{35}P_{56}$ consistent aggregation over time

very view indices are transitive; i.e. pass the circular test. Only Lowe and Cobb-Douglas

$$P_{0s}^{LW}P_{st}^{LW} = \frac{\sum p_s q}{\sum p_0 q}\frac{\sum p_t q}{\sum p_s q} = P_{0t}^{LW} = \frac{\sum p_t q}{\sum p_0 q}$$

$$P_{0t}^{CD} = \prod\left(\frac{p_t}{p_0}\right)^{\alpha_i} = \prod\left(\frac{p_1}{p_0}\right)^{\alpha_i}\prod\left(\frac{p_2}{p_1}\right)^{\alpha_i}\ldots\prod\left(\frac{p_t}{p_{t-1}}\right)^{\alpha_i}$$

1.2.2 (2) Second misunderstanding: Chaining (chainability)

Transitivity is very restrictive a property. It is implicitly assumed

1. proportionality with different basis (0, 1, or 2), or: different weights should <u>not</u> matter

$$P_{34} = \frac{P_{04}}{P_{03}} = \frac{P_{14}}{P_{13}} = \frac{P_{24}}{P_{23}}$$

2. Circularity is tantamount to the requirement that a certain matrix **P** of index numbers has to be a ***singular matrix***. **P** is defined as follows (in the case of T+1 = 4 rows and columns, t = 0, 1,..., T)

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix}$$

Transitivity implies identity $P_{tt} = 1$ and time reversibility ($P_{t0} = 1/P_{0t}$). Thus with T = 3 we have

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 1 & P_{01} & P_{01}P_{12} \\ 1/P_{01} & 1 & P_{12} \\ 1/P_{01}P_{12} & 1/P_{12} & 1 \end{bmatrix}$$

and the determinant $|\mathbf{P}|$ in fact vanishes. A consequence is that a single additional value, P_{23} is sufficient to calculate a fourth row and column (P_{03}, P_{13}, P_{23},P_{33}); although we do not even have to know which index formula is being used.

$$\mathbf{Pc} = \begin{bmatrix} 1 & P_{01} & P_{02} \\ 1/P_{01} & 1 & P_{12} \\ 1/P_{01}P_{12} & 1/P_{12} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ P_{23} \end{bmatrix} = \begin{bmatrix} P_{02}P_{23} \\ P_{12}P_{23} \\ P_{23} \end{bmatrix} = \begin{bmatrix} P_{03} \\ P_{13} \\ P_{23} \end{bmatrix} = \mathbf{p}$$

The misunderstanding reads as follows:

A **chain**-index makes consistent (transitivity, circular test) multi-period comparisons (aggregation over time) **by chaining** (that is, multiplying) successive two-period comparisons

[a comparison two *adjacent* periods is more legitimate and easier to carry out]

A chain-index is gained by **chaining**, however **not chainable**, but rather **path dependent** (the very opposite of transitivity)

1. Not only is a chain index different from the direct index

drift function →

$$\overline{P}_{0t}^{L} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \frac{\sum p_2 q_1}{\sum p_1 q_1} \cdots \frac{\sum p_t q_{t-1}}{\sum p_{t-1} q_{t-1}} \neq P_{0t}^{L} = \frac{\sum p_t q_0}{\sum p_0 q_0}$$

2. the chain indices for the same interval in time (0, t) are also different from one another depending on how the interval is partitioned into sub-intervals

the result differs also depending on the kind of subdivision (partitioning)
Chain indices therefore fail multi-period proportionality (and thus identity)

t = 0		t = 1		t = 2		t = 3		t = 4	
p	q	p	q	p	q	p	q	p	q
2	10	4	12	3	20	1	16	2	10
5	20	3	15	4	10	4	12	5	20

Note: the same prices and quantities in 0 and 4

$P_{04}^{L} = 1$

$\overline{P}_{04}^{L}(a) = P_{02}^{L} P_{24}^{L} = 0{,}825$

$\overline{P}_{04}^{L}(b) = P_{1}^{L} P_{2}^{L} P_{3}^{L} P_{4}^{L} = 0{,}7419$

other **chain** indices

Paasche: 1.212 = 1/0.825
Fisher: 1 (V-shape)

Paasche: 0.756
Fisher: 0.749

Lack of identity will also be examined later in **sec. 3.4.**

A chain index is **drifting** away from the direct index*

Determinants of drift (see later)

* see part IV for methods proposed to remove the "**chain drift**", or **lack of transitivity**

1.2.2 (5) Chaining, chainability and superiority of chain indices

Peter Hill* gave an interesting interpretation of the following inequation

$$\overline{P}_{0t}^{L} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \frac{\sum p_2 q_1}{\sum p_1 q_1} \cdots \frac{\sum p_t q_{t-1}}{\sum p_{t-1} q_{t-1}} \neq P_{0t}^{L} = \frac{\sum p_t q_0}{\sum p_0 q_0}$$

"...it must also be asked whether it is reasonable to judge a chain index by compar-ing it with its direct counterpart“ and "Advocates of chaining ought not to be in favour of circularity be-cause the identity between direct and indirect comparisons which satisfaction of the circularity test ensures makes the construction of a chain index superfluous. On the contrary, there must actually be a difference between the direct and the indirect measure for the latter to be superior on some criterion."

Logic: in the absence of a specified criterion the simple fact that an index formula P*, however absurd it may be, deviates von P^L can be taken as a proof that P* is superior.

$P_{0t}^{*} \neq P_{0t}^{L}$ a blessing?

Moreover **chaining** [the operation] is o.k.
but **chainability** [the idea justifying this operation] is not desirable.

* Index Number Theory and External Trade, Eurostat News-Special Edition, Proceedings of a Seminar held in Luxembourg 1988

1.2.2 (6) Digression: more about drift (see also later sec. 3.4)

1. Notion of "drift": Example: definition of the Laspeyres, price index drift (or the Paasche quantity index drift)

$$D_{0t}^{PL} = \overline{P}_{0t}^{L} / P_{0t}^{L}$$

$$D_{0t}^{QP} = \overline{Q}_{0t}^{P} / Q_{0t}^{P}$$

2. Theory about the drift function (i.e. determinants of drift)

$$D_{02}^{PL} = \frac{Cov(x_{12}, y_{01})}{\bar{x}_{12} \cdot \bar{y}_{01}} + 1$$

Note the cumulative structure of the drift function

$$D_{03}^{PL} = D_{02}^{PL}\left(\frac{Cov(x_{23}, y_{02})}{\bar{x}_{23} \cdot \bar{y}_{02}} + 1\right) \text{etc.}$$

The drift functions depends (much like the chain index function to which they refer) on the length of the interval (0, t) in question, on how it is subdivided into subintervals, and on the path (pattern of the p's and q's).

where

$$x_{i,12} = \frac{p_{i2}}{p_{i1}}, x_{i,23} = \frac{p_{i3}}{p_{i2}}, \ldots \text{ (links)}$$

$$y_{i,01} = \frac{q_{i1}}{q_{i0}}, y_{i,02} = \frac{q_{i2}}{q_{i0}}, \ldots \text{ (relatives)}$$

More about drift in section **3.5**

1.2.3 (1) **Third misunderstanding**: direct index a special case of chain index

As a product of **continually updated links** the chain-index is a more general concept. Multiplication of links facilitates adaption to new conditions and accounting for new/disappearing goods

A direct index has a product representation too (different however)*

$$P_{03}^{L} = \left(\sum \frac{p_1}{p_0} \frac{p_0 q_0}{\sum p_0 q_0}\right)\left(\sum \frac{p_2}{p_1} \frac{p_1 q_0}{\sum p_1 q_0}\right)\left(\sum \frac{p_3}{p_2} \frac{p_2 q_0}{\sum p_2 q_0}\right) = \sum \frac{p_3}{p_0} \frac{p_0 q_0}{\sum p_0 q_0}$$

$$\overline{P}_{03}^{L} = \left(\sum \frac{p_1}{p_0} \frac{p_0 q_0}{\sum p_0 q_0}\right)\left(\sum \frac{p_2}{p_1} \frac{p_1 q_1}{\sum p_1 q_1}\right)\left(\sum \frac{p_3}{p_2} \frac{p_2 q_2}{\sum p_2 q_2}\right)$$

partial up-date of weights (of prices only)

The direct index seems to be a special case of the chain index, in that only prices are updated (and a somewhat incomplete and inferior special case)

this raises some questions → part II

* disregarding the change of the domain of definition

1.2.3 (2) The role of multiplication: direct Laspeyres and price updating

Note that the factors on the right hand side (RHS) of the second equation are *not* the "ordinary" Laspeyres indices, but a ***sequence of rebased Laspeyres indices***

$$P_{12(0)} = \frac{P_{02}}{P_{01}} = \frac{\sum p_2 q_0}{\sum p_1 q_0} \qquad P_{23(0)} = \frac{P_{03}}{P_{02}} = \frac{\sum p_3 q_0}{\sum p_2 q_0} \qquad \text{etc.}$$

on the other hand

$P_{t-1,t\,(0)}$ is just the price updated Laspeyres link

$$P_t^{L(p)} = \frac{\sum p_t q_0}{\sum p_{t-1} q_0} = \sum \frac{p_t}{p_{t-1}} \left(\frac{p_{t-1} q_0}{\sum p_{t-1} q_0} \right)$$

A common criticism of the direct Laspeyres is that *weights* q_0 in become *progressively irrelevant* with the passage of time. In the same vein in t weights q_{t-1}, q_{t-2}, q_{t-3} should also be (in this order) considered as "progressively irrelevant". **Why not delete** those **obsolete weights** ? Which (quantity) weights are involved? (and which should be deleted)

strictly speaking: the notion "always most recent weights" would best apply to direct Paasche

t	P_{0t}^L	P_{0t}^P	$\overline{P}_{0t}^L$	$\overline{P}_{0t}^F$
1	q_0	q_1	q_0	q_0, q_1
2	q_0	q_2	q_0, q_1	q_0, q_1, q_2
3	q_0	q_3	q_0, q_1, q_2	q_0, q_1, q_2, q_3
4	q_0	q_4	q_0, q_1, q_2, q_3	q_0, q_1, q_2, q_3, q_4

1.2.3 (3) Third misunderstanding: multiplication mystery and flexibility

A direct index

- can be written in both ways, as a ratio **and** a product, chain indices, however, can **only** be written (and compiled) as a product
- provides a **pure price comparison** (unlike chain indices)

The flexibility of chain indices is owed to the fact that the link-function is constantly **changing** its **domain of definition** by contrast to direct superlative indices such as Fisher, Törnquist …

$$\frac{\sum_i p_{1i} q_{0i}}{\sum_i p_{0i} q_{0i}} \frac{\sum_k p_{2k} q_{1k}}{\sum_k p_{1k} q_{1k}} \frac{\sum_m p_{3m} q_{2m}}{\sum_m p_{2m} q_{2m}}$$

The result of a chain index is reflecting

- the change of prices (for the same goods),
- change of weights (quantities) {accounting for substitution}
- the path connecting 0 and t (path dependence)
- the changing domain of definition

1.3 The increasing relevance of chaining: Scanner data

Scanner data provide

1. information on prices in much greater detail
2. at a much higher frequency
3. in combination with quantity data on the level of individual products*

There are, however, more problems with

1. ensuring pure price comparison, as (such frequent) indices are necessarily chained
2. time aggregation (unit values over weeks, months etc.)* and aggregation over outlets
3. sales can generate erratic movements of chained indices

* previously only infrequent and less detailed expenditure data derived from HES were available for weighting

* according to IFD "time aggregation choices … have a considerable impact on estimates of price change"

R. J. Hill : "the often erratic behavior of chained price indexes in scanner data sets"

Ivancic, Fox, Diewert (IFD) report: When chained indexes are used, the difference in price change estimates can be huge, ranging from minus 1.42% to minus 25.78% for a superlative (Fisher) index and an incredible **17.22% to 9,548%** for a non-superlative (Laspeyres) index. The results suggest that traditional index number theory breaks down when weekly data with severe price bouncing are used, even for superlative indexes …quarterly indices are largely free of drift" *(on their methods to deal with "chain drift" → part IV)*

2.1/2 These sections present the most frequently presented **arguments in favour of chain indices**. An attempt is made to give a *systematic* account and critique of them.

There are in principle two hardly refutable arguments

- chain indices approximate superlative indices (smaller Laspeyres-Paasche gap) (= D2)*
- with chain indices there are less problems with matching and quality adjustment (= C2)*

The first argument is being discussed in more detail in sec. 2.3

2.3 **Laspeyres-Paasche-gap (LPG,** also known as Paasche-Laspeyres **Spread PLS**): this section reviews theories and empirical finding about the conditions under which the two chain indices will differ less than the respective direct indices

* D2/C2 refers to our systematic overview over arguments in favour of chain indices

2.1 (1) Twelve arguments in favour of chain indices, an overview

Arguments focussing on

- **A**: links
  - **A1:** why not accelerate
  - **A2:** only valid information
  - **A3:** multiplication mystery*
- **B**: the "base"
  - **B1:** moving comparison
  - **B2:** independent of base
  - **B3:** growth factor
- **C**: weights
  - **C1:** updating of weights
  - **C2:** new developments
    - Less problems with new/disappearing goods
- **D**: results
  - **D1:** accounts for substitution, smooth, less inflation
    - Approximates COLI
  - **D2:** less formula problems
  - **D3:** goodness of fit (models)
- **E**: deflation
  - **E:** SNA recommendations → see part II

* this argument also comprises the idea that chain indices provide valuable **additional information** because of making better use of all time series data

2.1 (2) General characteristics of the twelve arguments in favour of chain indices

1. justification of chain indices is **not theory-driven** (e.g. COLI is a new theory)
2. "advantages" of chain indices are mainly derived from a **critique of the *fixed basket*** (direct *Laspeyres*) **approach** (e.g. weights are also updated in *direct* superlative indices [using q_0 and q_t]). However:

   chain indices are <u>not</u> recommended

   - when comparisons over *long* intervals in time rather than short ones are wanted
   - consumption patterns change *rapidly* and *fundamentally* rather than smoothly in response to changes in prices (and just these are the situation in which P^L might fail)

3. **problems** purportedly "solved" by chain indices are not really solved but **rather "dissolved"**

   example: choice of base year, problems with quality adjustment

4. occasionally **inconsistent** and *inconclusive* statements; e.g. the SNA (93)

   unit value indices are "affected by changes in the mix of items as well as by changes in their prices. Unit value indices cannot therefore be expected to provide good measures of average price change over time" (§ 16.13)

5. **playing down of counter-arguments**, e.g. non-additivity, path dependence and not much attention is given to **problems of data collection** and cost in official statistics (updating a more or less detailed structure of weights)

2.2.1 (1) Arguments class A: Focussing on the element "link"*

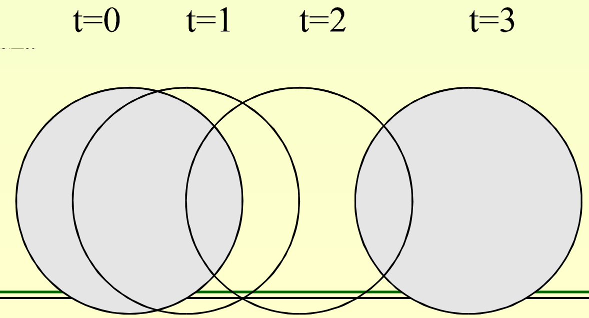
* and disregarding the existence of *two* elements , link and chain

General: claiming an advantage of chain indices arising from the simple fact that the interval (0, t) is subdivided into sub-intervals and the index is derived from multiplying links

Argument	Rebuttal
A1 "**why not**", "**limiting case**" "Chaining is merely the limiting case where the base is changed each period" "In effect, the underlying issue is not whether to chain or not but how often to rebase. Sooner or later the base year for fixed weight Las-peyres ... indices ... has to be up-dated" (SNA93 §16.77) "why not accelerate and go for annual chaining? There is no reason why not." (Allen) **The reason is pure comparison and no path dependence**	The problem is not frequent rebasing (the base of links is always t-1) but multiplication of links. A1 is a **misinterpretation of P^L**: The guiding principle of the fixed-basket-approach is comparability within an interval rather than across intervals In general direct indices referring to different base periods will not be multiplied. In the chain approach links are *necessarily* chained together and (unlike direct indices) chain indices are path dependent. **Consider rebasing at t = 5** P^L_{09} has 3 **p** vectors ($\mathbf{p}_0$, $\mathbf{p}_5$, $\mathbf{p}_9$) and 2 **q** vectors ($\mathbf{q}_0$, $\mathbf{q}_5$) $\bar{P}^L_{09}$ has 10 **p** vectors and 9 **q** vectors **If annual chaining is better, why not monthly?**

2.2.1 (2) Arguments A: Focussing on the element "link"*

Argument	Rebuttal
A2 "only valid information", the only validly obtainable information is the direction of *change* from *year to year*, not the *level* over a *long* period. Or: good because link is short (A3: good because chain is long) (both A2 and A3 arguments of Mudgett)	If A2 were correct we should rather refrain from multiplying links to a chain. A weak link is able to weaken the whole chain while each value of a direct index is an independent estimate on its own. Why is P^{L}_{09} not valid because of the long distance between 0 and 9 and $\overline{P}^{L}_{09}$ is valid?
A3 multiplication mystery 1) a valid procedure for making comparisons over *long* series or distant areas by multiplication of those links (Mudgett) 2) Chain indices are making better use of the information in a time series and provide useful additional information	Direct contradiction to A2. No proof for the use of "additional information" given Why things which are directly not com-parable are so indirectly? However, the *logical status of the comparison* is different. see also argument **B1** and **C2**



2.2.2 (1) Arguments B: Ambiguities concerning the notion "base"

General: the "base" to which a time series of indices or of year-to-year growth rates refers is more relevant and realistic

Argument	Rebuttal
B1 Chain indices provide a different type of comparison by making use of a **"moving" base***	The base of the link is moving, not the base of the chain The problem "choice of a base" (of a chain) is not "solved", but rather made irrelevant** (once 0 is given weights are uniquely determined) Why is $\overline{P}_{01}^{L}, \overline{P}_{02}^{L}, ..., \overline{P}_{0t}^{L}$ a "run" and $P_{01}^{L}, P_{02}^{L}, ..., P_{0t}^{L}$ is not a run? The fundamental difference: one is path dependent, the other is not.

The additional information argument rests on the assumption

if there are more data (reflecting more phenomena) entering a formula, i.e. **more *data input***

→

then the resulting statistic is more "informative", that is then we also get **more *information output***

* Chains are said to be "*runs*" of index numbers instead of binary comparisons only and they allegedly provide *valuable additional information*

** The value of an index in t is no longer expressed "in percent (in units) of the base period value"

2.2.2 (2) Arguments B2 + B3: Ambiguities concerning the notion "base"

Argument	Rebuttal
B2 Chain indices are **independent of the base** (or: they have "no base", or: the base is always t-1)	The **reference base** (0) is irrelevant, P_{34} is the same irrespective of the base $\overline{P}_{04}/\overline{P}_{03}=\overline{P}_{14}/\overline{P}_{13}=\overline{P}_{24}/\overline{P}_{23}=P_4$ see sec. **3.2** for the implicit assumptions in this equation In chain indices the ***reference*** **base** (**RB** =100) is deemed irrelevant. On the other hand it is the increased attention given to the ***weight*** **base** (**WB**) and its up-to-dateness.
B3 More relevant **growth factor**	No attempts made to quantify the extent to which weights are more "relevant", "realistic" or "representative"

	Growth of Norwegian GDP	1987	1988	1989
A	constant base period prices*	**4.9**	**3.0**	**5.2**
B	previous year prices	**3.9**	**1.8**	**0.9**

* fixed prices of 1984

The reason for this situation seems to be that oil prices in 1984 were much higher than in 1987 and in particular in 1988.

2.2.2 (3) Argument B3 (viewed as most important by Eurostat etc.)

index*	Good growth rate	Bad growth rate
Price index	$\frac{\bar{P}_{0t}^{LC}}{\bar{P}_{0,t-1}^{LC}} = P_t^{LC} = \frac{\sum p_t q_{t-1}}{\sum p_{t-1} q_{t-1}}$	$\frac{P_{0t}^{L}}{P_{0,t-1}^{L}} = \frac{\sum p_t q_0}{\sum p_{t-1} q_0}$
Quantity index (volumes)	$Q_t^{LC} = \frac{\sum q_t p_{t-1}}{\sum q_{t-1} p_{t-1}}$	$\frac{Q_{0t}^{L}}{Q_{0,t-1}^{L}} = \frac{\sum q_t p_0}{\sum q_{t-1} p_0}$

constant weights **comparable** over time

see also sec. **3.2** for the equation: most recent = most important (or relevant)

The difference as regards the relevance or irrelevance of RB and WB respectively, gives rise to the questions:

1. What makes the choice of the base period difficult in the direct index framework?
2. Is it possible to choose a "wrong" (in-adequate) base in the chain index framework?

base in the direct index framework

the price level in period t measured ***in terms of*** the level in 0, or

the value of P^L in t is expressed "***in percent***" ***(in units) of*** the base period value". Irrelevance of the RB then is anything but desirable.

* the argument B3 does **not apply to** the **Paasche** formula

2.2.2 (4) Digression: growth rates of monthly chain indices

Surprisingly in annual growth rates of figures compiled monthly we already have **two** quantity structures for example q_{08} , and q_{09} influencing the result

Prices	Jan. 2009	...	May 2009	...	Dec. 2009	Jan. 2010
Weights	∅ 2008	...	∅ 2008	...	∅ 2008 ↔	∅ 2009

↕

switch of weights takes place in December

Prices	Jan. 2008	...	May 2008	...	Dec. 2008	Jan. 99
Weights	∅ 2007	...	∅ 2007	...	∅ 2007 ↔	∅ 2008

The problem would have been avoided if at the end of the year 2008 all monthly 2008 price indices were re-calculated using weights of 2008.

(in the same manner: at the end of 2009 re-calculation using weights of 2009)

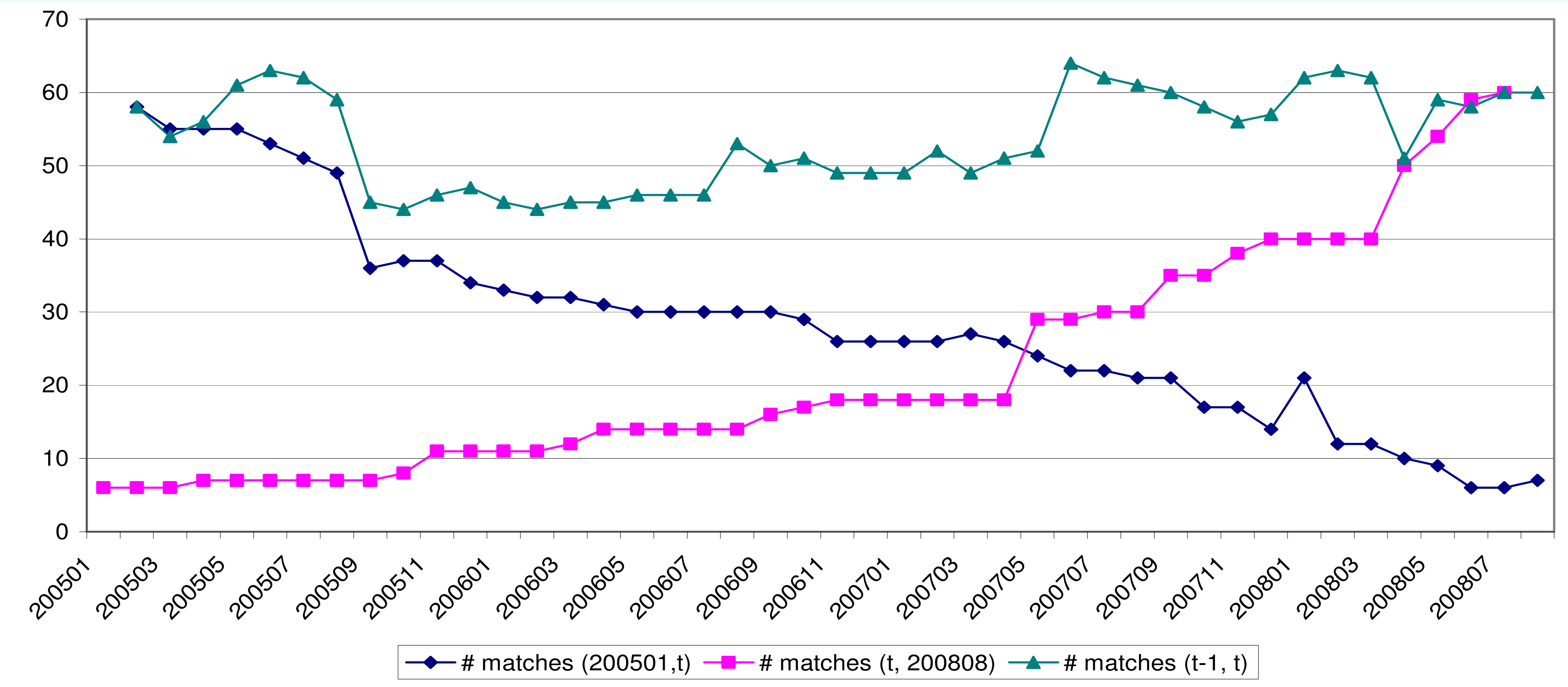
2.2.3 (1) Arguments class C: Flexibility and the continually updated weights

General: superior flexibility and adaptability as regards the structure of weights and the appearance of new and disappearance of old goods;
(chaining better fits to our modern times and is an elegant device to elude the trouble with keeping a basket and the sample of outlets constant over time)

Argument	Rebuttal
C1 Most frequent update of weights; SNA93, § 16.41 "indices whose weighting structures are as up-to-date and relevant as possible"	1) This argument again compares a direct index with a chain index (or **rather a link** only), as if they both had a single weighting scheme only. 2) There is no clear concept or measure of the *degree* of "representativity" or "relevance". 3) Given that a price index ought to reflect new *quantity* weights, what then is the task of a *quantity* index?
C2 Less problems with **new developments**, and **quality adjustment** is less difficult problem of **matching** (an acceptable argument)	It is right that the fixed basket approach of the Laspeyres direct index inevitably (and increasingly) runs into difficulties as new products emerge, old ones are no longer available. The traditional solution was: quality **adjustments, imputations** etc. However, C2 amounts to **giving up** the aim "**pure price comparison**"

2.2.3 (2) Argument C2 (problems with pure price comparison)

Number of matched items; detergents*

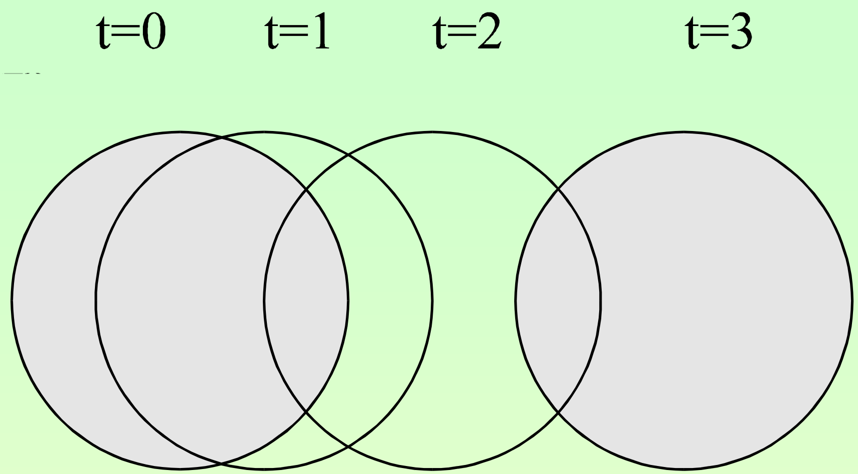


* Jan de Haan and Heymerik van der Grient: Eliminating Chain Drift in Price Indexes Based on Scanner Data 15 (September 2009)

The downward sloping curve shows:
Only seven out of the 58 initial items (January 2005) can still be purchased at the end of the period (August 2008). Hence **adhering to a strict matched-item principle** (using a completely fixed sample of items for the sake of pure price comparison) is **impossible** (or requires many imputations)

2.2.3 (3) More about some arguments: C2 and A3

C2 is a bit similar to the multiplication mystery **A3** (what is directly in-comparable nonetheless becomes comparable in an indirect approach)



successive comparisons of only partially overlapping circles relate 0 to 3

"Dissolution" of a problem:

No longer aiming at a pure price comparison (over more than just two adjacent periods): As the basket is allowed to (or even bound to) change constantly there is no point in taking care for comparability of the basket in t with the basket in 0

The increase in convenience has to be contrasted, with the fact that

- chain indices require more resources for empirical studies needed for the up-dating of weights, and that
- comparability over more than two adjacent periods, is relaxed if not abandoned.

2.2.4 (1) Arguments class D: Results, approximation of superlative indices*

General: The focus here is on expected favourable numerical results when chain indices are to be used. The arguments do not refer to conceptual aspects and apply to *all* sorts of empirical data, like for example axiomatic considerations.

Argument	Rebuttal
D1 Smoother development, **less inflation**	Low inflation is likely primarily "if individual prices and quantities tend to increase or decrease monotonically over time" (SNA 93, § 16.44).* Severe problems with chain indices in the case of oscillating prices
D2 Less **choice of formula** problems	Argument rests on often observed smaller **Laspeyres-Paasche-gap** (**LPG**) see more in → **2.3** "the choice of index number formula assumes less significance" (SNA 93, § 16.51, similar in CPI Manual) It is tacitly assumed (and contentious) P^L and P^P are "equally plausible" or "equally justifiable".
D3 Goodness of fit in econo-metric models	Typically brought forward on the part of the "stochastic approach" A higher goodness of fit in a regression model (whatsoever) is taken as a proof of *conceptual* superiority

* Interestingly these are precisely those conditions under which P^L is not that bad.

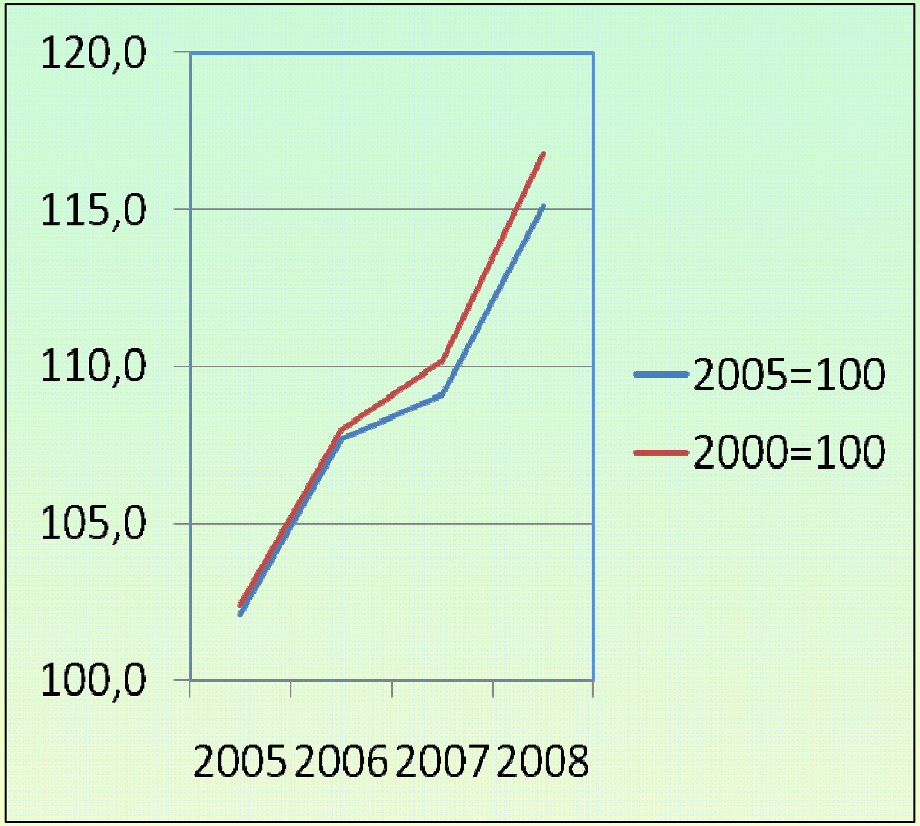
2.2.4 (2) More about arguments D1/2: smoothness, less inflation, closer to Fisher

D1: less inflation

Index with a more recent base tends to be lower

Example German PPI

(WiSta 8/2009, p. 813)



D2: Choice of formula less relevant; **approaching Fisher's** ideal **index** (or other **superlative indices**)

If this is the main motivation (e.g. Paul Schreyer, OECD) questions arise:

Why not take **a direct superlative index** (Fisher, Törnquist, Walsh): where there is no substitution bias by definition?

How to explain violation of identity* or chain drift with substitution bias?

* Slide **18** (sec. **1.2.2. (4)**)

2.2.4 (3) Are chain indices approximating "superlative" Indices?

Superlative (= ^{sup}) indices can be expressed as "quadratic means" (= geometric means of weighted [using expenditure shares s] "power means" $P_{r/2}$) are

$$P_{0t}^{sup} = \sqrt{P_{r/2}P_{-r/2}} = \sqrt{\left(\sum s_{i00}\left(\frac{p_{it}}{p_{i0}}\right)^{r/2}\right)^{2/r}\left(\sum s_{itt}\left(\frac{p_{it}}{p_{i0}}\right)^{-r/2}\right)^{-2/r}}$$

or equivalently $P_{0t}^{sup} = \left(\sum s_{i00}\left(\frac{p_{it}}{p_{i0}}\right)^{r/2}\right)^{1/r}\left(\sum s_{itt}\left(\frac{p_{it}}{p_{i0}}\right)^{-r/2}\right)^{-1/r}$

Superlative* indices are those "that are exact (i.e. equal to the cost of living index) for flexible expenditure functions (i.e. … that are twice continuously differentiable and can approximate an arbitrary linearly-homogenous function to the second order)" (Hill, p. 312)

Special cases: $r \to 0$ (Törn-quist), r = 2 Fisher, r = 1 implicit Walsh

$$\sqrt{P_{2/2}P_{-2/2}} = \sqrt{\left(\sum s_{i00}\left(\frac{p_{it}}{p_{i0}}\right)\right)\left(\sum s_{itt}\left(\frac{p_{it}}{p_{i0}}\right)^{-1}\right)^{-1}} = \sqrt{P_{0t}^{L}P_{0t}^{P}} = P_{0t}^{F}$$

For most data sets T, F and W approximate each other closely

* according to Diewert (1978)

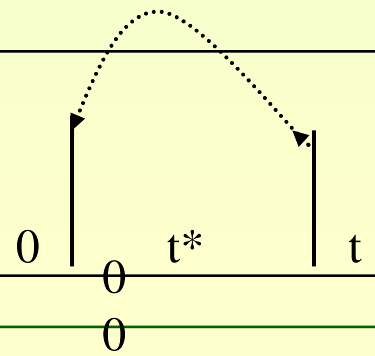
2.2.4 (4) **Not** approaching superlative Chain drift and "path dependence" of chaining

As chain may be within the P^P-P^L interval they are viewed as approximating Fisher's "ideal" index. However **path dependence** may well lead to chain indices > P^L, or < P^P respectively (more in sec. 2.3)

Path dependence of chain indices and its determinants are well known facts

The **SNA 93** (§ 16.47 – 49) states that a chain index should

- ***not*** be used when prices are ***cyclically*** moving (rising and declining, and thereafter returning to a certain level in some regular manner) by contrast to
- a (***moderate***) ***monotone*** rise or decline of prices, in which case a chain index is recommended, or in summary SNA 93 arrived at the following rule:

when relative prices in the first and the last periods (0,t) are		a chain index
1) **very different from each other** and chaining involves linking periods in which prices and quantities are intermediate between those of 0 and t		should be used
2) similar to each other (and very different to an intermediate period t*); example: seasonal variation		should **not** be used: no indirect comparison via t*

2.2.5 Group E arguments: advantages in deflation (position of the SNA 93)

SNA recommendations

1. the preferred measure of year to year movement of **real GDP** is a **Fisher volume index**, changes over longer periods being **obtained by chaining**: that is, by cumulating the year to year movements;

2. the preferred measure of year to year **inflation** for GDP is therefore a **Fisher price index**, price changes over long periods being **obtained by chaining** the year to year price movements: the measurement of inflation is accorded equal priority with the volume measurements;

3. chain indices that use **Laspeyres volume** indices to measure movements in real GDP and Paasche price indices to year to year inflation provide acceptable alternatives to Fisher indices

are the price indices of NA really inflation measures?

Some necessary remarks

1) Fisher index (even as direct index) is far from being ideal (factor reversal test)
2) Non-additivity already well known at the time of the SNA 93
3) More disadvantages (I only after publishing my monograph "Chain Indices" became aware of): QNA-ANA-consistency: **More about 1 – 3 in parts II and III** respectively

2.3 Laspeyres-Paasche-Gap LPG (or: Paasche- Laspeyres-Spread PLS)

For some NSIs* the **reduced LPG** was one of the **most important advantage of chaining** in their decision to move from direct Laspeyres to chained Laspeyres (e.g. for Australian Bureau of Statistics)

More about LPG in sec. 3.5 (drift)

Definitions: *NSI = National Statistical Institute

	direct indices	**chain** indices
LPG	$\gamma_{0t} = P_{0t}^{L} - P_{0t}^{P}$	$\bar{\gamma}_{0t} = \bar{P}_{0t}^{L} - \bar{P}_{0t}^{P}$ ← For chains of any length
PLS* (Hill)	$PLS_{13}^{D} = \frac{\max\left(P_{13}^{P}, P_{13}^{L}\right)}{\min\left(P_{13}^{P}, P_{13}^{L},\right)}$	$PLS_{13}^{C} = \frac{\max\left(P_{12}^{P}, P_{23}^{P}, P_{12}^{L}, P_{23}^{L}\right)}{\min\left(P_{12}^{P}, P_{23}^{P}, P_{12}^{L}, P_{23}^{L}\right)}$

* The definition can also be used for more than two indices, e.g.

$$PLS_{13}^{D} = \frac{\max\left(P_{13}^{F}, P_{13}^{T}, P_{13}^{W}\right)}{\min\left(P_{13}^{F}, P_{13}^{T}, P_{13}^{W}\right)}$$

Theory:

To date there is still no general theory of the LPG/PLS

Robert J. Hill challenged **the general belief that chaining reduces** (increases) the **PLS whenever prices and quantities are monotonic** (fluctuating) over time. However, he also found that monotonicity (defined in various ways) is neither necessary nor sufficient to ensure a reduction of the PLS.

2.3.1 (1) Laspeyres-Paasche-Gap: Monotonous movement of prices

1) The following slides provide a simple situation in which in fact holds:

$$P_{0t}^{P} < \overline{P}_{0t}^{P} < \overline{P}_{0t}^{F} \approx P_{0t}^{F} < \overline{P}_{0t}^{L} < P_{0t}^{L}$$

To this end we assume constant growth rates of both, prices as well as quantities of two commodities, A and B

2) We then will **slightly modify the assumptions** concerning the (constantly declining) quantities [negative covariance in the Bortkiewicz formula] and we will get

$$P_{0t}^{P} > \overline{P}_{0t}^{P} > \overline{P}_{0t}^{L} > P_{0t}^{L}$$

This situation is difficult to interpret in terms of a "substitution bias"

In *both* cases prices and quantities change monotonously over time and chaining therefore reduces the LPG. By contrast section 2.3.2 demonstrates – just as theory suggests – that chaining increases the gap between (chained as opposed to direct) P^L and P^P when prices and quantities fluctuate (or "bounce").

2.3.1 (2) A thought experiment: wide divergence between Laspeyres and Paasche (1)

Constant changes of prices and quantities

prices

$\omega_1 = p_{1t}/p_{1,t-1} = \mathbf{1.1} \quad \omega_2 = p_{2t}/p_{2,t-1} = \mathbf{1.2}$

quantities

$\lambda_1 = q_{1t}/q_{1,t-1} = \mathbf{0.9} \quad \lambda_2 = q_{2t}/q_{2,t-1} = \mathbf{0.8}$

assumptions concerning λ will be modified later

$p_{10}q_{10} = p_{20}q_{20} = 50$

Quantity changes seem to off-set price changes. P^L does not depend on the λ terms (quantity relatives).

$$P^L_{0t} = \frac{1}{2}\left(\omega_1^t + \omega_2^t\right)$$

P^L is not depending on λ

	PL	L-ch	PF	F-ch	P-ch	PP
0	100,00	100,00	100,00	100,00	100,00	100,00
1	115,00	115,00	114,85	114,85	114,71	114,71
2	132,50	132,16	13182	131,66	131,15	131,15
3	152,95	151,78	151,20	150,82	149,86	149,48
4	176,89	174,20	173,32	172,65	171,12	169,84
5	204,94	199,79	198,56	197,52	195,27	19238
6	237,88	229,00	227,34	225,82	222,68	217,27
7	276,59	262,29	260,16	258,00	253,78	244,69
8	322,17	300,23	297,56	294,58	289,03	274,83
9	375,89	343,42	340,19	336,12	328,97	307,89
10	439,27	392,58	388,78	383,27	374,19	344,09

PL = direct Laspeyres. PP = direct Paasche, PF = direct Fisher
L-ch = Laspeyres chain, P-ch = Paasche chain, F-ch = Fisher ch.

2.3.1 (3) Experiment: wide divergence between Laspeyres and Paasche $P^L > P^P$

direct Paasche

$$P_{0t}^{P} = \omega_1^t g_{1t} + \omega_2^t g_{2t}$$

where $g_{1t} = (\lambda_1)^{t-1} / \left[(\lambda_1)^{t-1} + (\lambda_2)^{t-1}\right]$ and $g_{2t} = 1 - g_{1t}$

If $\omega_1 = \omega_2 = \omega$ then all indices equal

$$P_{0t}^{L} = \omega^t$$

chain Laspeyres (link)

$$P_{t-1,t}^{L} = \frac{\sum p_t q_{t-1}}{\sum p_{t-1} q_{t-1}} = \omega_1 g_{1t}^{*} + \omega_2 g_{2t}^{*}$$

If $\lambda_1 = \lambda_2 = \lambda$ then all indices equal

$$P_{0t}^{L} = \tfrac{1}{2}\left(\omega_1^t + \omega_2^t\right)$$

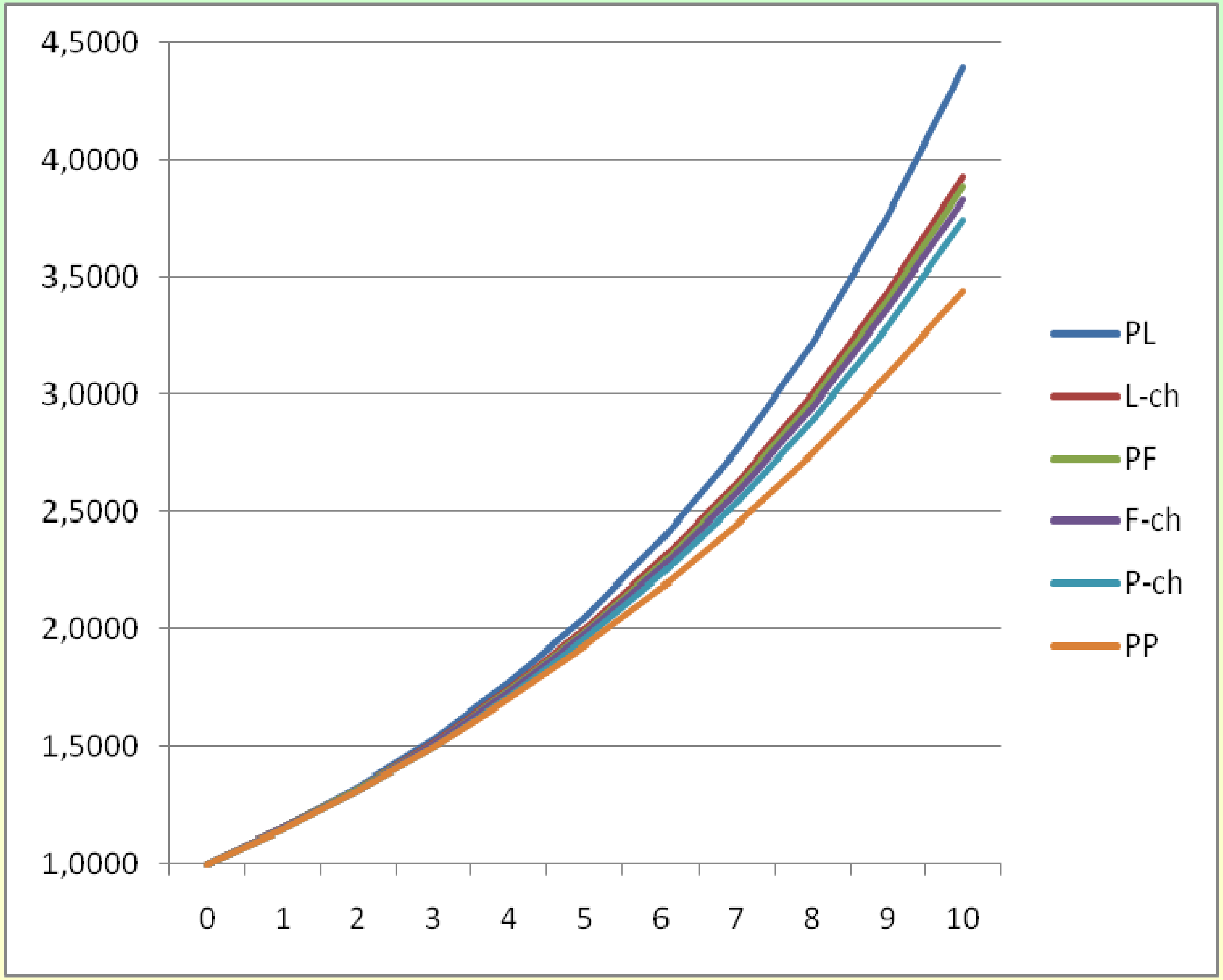
where $g_{1t}^{*} = (\lambda_1\omega_1)^{t-1} / \left[(\lambda_1\omega_1)^{t-1} + (\lambda_2\omega_2)^{t-1}\right]$ and $g_{2t}^{*} = 1 - g_{1t}^{*}$

chain Paasche (link)

$$P_{t-1,t}^{P} = \frac{\sum p_t q_{t-1}}{\sum p_{t-1} q_{t-1}} = \omega_1 g_{1t}^{**} + \omega_2 g_{2t}^{**}$$

where $g_{1t}^{**} = \lambda_1 (\lambda_1\omega_1)^{t-1} / \left[\lambda_1 (\lambda_1\omega_1)^{t-1} + \lambda_2 (\lambda_2\omega_2)^{t-1}\right]$ and $g_{2t}^{*} = 1 - g_{1t}^{*}$

2.3.1 (4) A thought experiment (the expected results concerning LPG)



Chain indices approximate the super-lative direct Fisher index.

The gap is constantly widening: This applies also to the growth rates

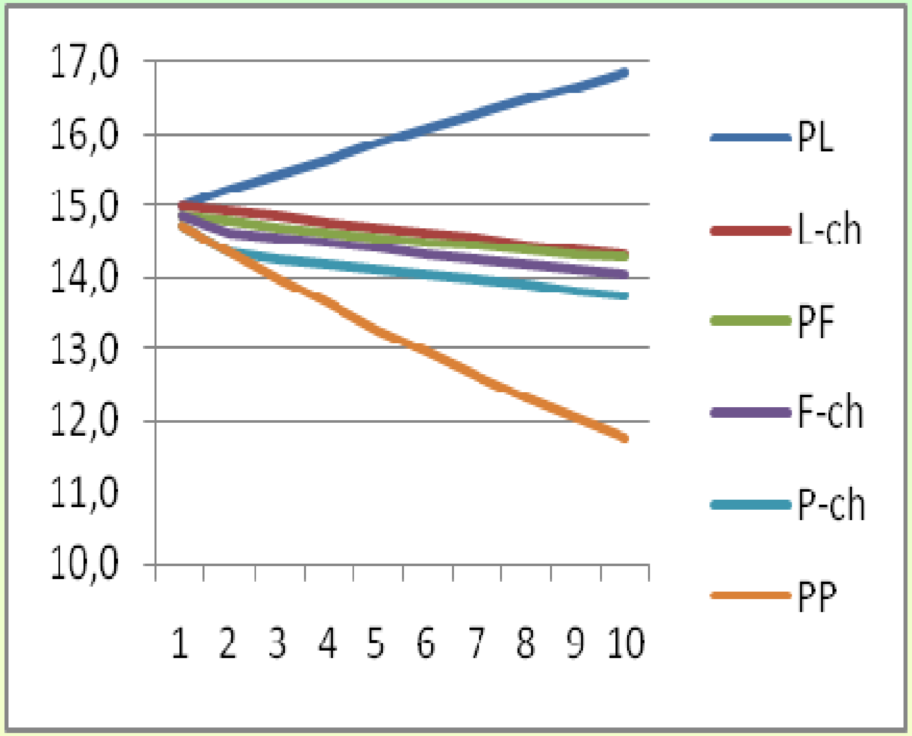
2.3.1 (5) The experiment (concept of LPG applies): growth rates

growth rates of the example
a monotonous development

	PL	L-ch	PF	F-ch	P-ch	PP
1	15,0	15,0	14,9	14,9	14,7	14,7
2	15,2	14,9	14,8	14,6	14,3	14,3
3	15,4	14,8	14,7	14,6	14,3	14,0
4	15,6	14,8	14,6	14,5	14,2	13,6
5	15,9	14,7	14,6	14,4	14,1	13,3
6	16,1	14,6	14,5	14,3	14,0	12,9
7	16,3	14,5	14,4	14,3	14,0	12,6
8	16,5	14,5	14,4	14,2	13,9	12,3
9	16,7	14,4	14,3	14,1	13,8	12,0
10	16,9	14,3	14,3	14,0	13,7	11,8

↓ 20 (PL) ↓ 10 (PP)

The pattern of growth rates shows:
The gap is widening



It can be shown: The growth rate (factor) of P^L tends to the higher of the two price relatives (ω_2 = 1.2). Likewise (more difficult to show):

The growth rate (factor) of P^P tends to the lower of the two price relatives (ω_1 = 1.1).

2.3.1 (6) The experiment (concept of LPG applies): growth rates

The only index having a constant growth rate as a geometric mean of ω_1 and ω_2 is the (transitive) Cobb-Douglas index (v.d.Lippe, 2007, p. 230)

The **convergence of growth factors** to the higher/lower ω is in some cases easy to show

1) direct Laspeyres

$$\frac{P^L_{0t}}{P^L_{0,t-1}} = \frac{\omega_1^t + \omega_2^t}{\omega_1^{t-1} + \omega_2^{t-1}} = \omega_2 - \frac{\omega_2 - \omega_1}{1 + \left(\frac{\omega_2}{\omega_1}\right)^{t-1}}$$

as $\omega_2 > \omega_1$ it is easy to see that

$$\lim_{t\to\infty}\left(\frac{P^L_{0t}}{P^L_{0,t-1}}\right) = \omega_2$$

an equivalent equation is

$$\frac{P^L_{0t}}{P^L_{0,t-1}} = \omega_1 - \frac{\omega_1 - \omega_2}{1 + \left(\frac{\omega_1}{\omega_2}\right)^{t-1}}$$

2) chain Paasche another simple relation

Paasche link

$$P^P_{t-1,t} = \omega_1 \frac{1}{1 + \frac{\lambda_2(\lambda_2\omega_2)^{t-1}}{\lambda_1(\lambda_1\omega_1)^{t-1}}} + \omega_2 \frac{\frac{\lambda_2(\lambda_2\omega_2)^{t-1}}{\lambda_1(\lambda_1\omega_1)^{t-1}}}{1 + \frac{\lambda_2(\lambda_2\omega_2)^{t-1}}{\lambda_1(\lambda_1\omega_1)^{t-1}}} = \omega_1 G_{1t} + \omega_2 G_{2t}$$

Weight of ω_1 in the example

t	G_{1t}
1	0.5371
2	0.5447
3	0.5523
4	0.5599

G_1 is constantly rising giving more and more weight to ω_1

2.3.1 (7) A slight modification of the experiment: Laspeyres < Paasche

Again constant changes of prices and quantities

prices *(as before)*

$\omega_1 = p_{1t}/p_{1,t-1} = \mathbf{1.1}$ $\omega_2 = p_{2t}/p_{2,t-1} = \mathbf{1.2}$

quantities *(modified)*

$\lambda_1 = q_{1t}/q_{1,t-1} = \mathbf{0.8}$ $\lambda_2 = q_{2t}/q_{2,t-1} = \mathbf{0.9}$

Modification: We simply interchanged λ_1 and λ_2

as before $p_{10}q_{10} = p_{20}q_{20} = 50$

	PL	L-ch	P-ch	PP
0	100.00	100.00	100.00	100.00
1	115.00	115.00	115.294	115.294
2	132.50	132.84	133.51	133.848
3	152.95	154.103	155.877	156.42
4	176.89	179.513	182.664	183.934
	exactly like before	> P^L		> P-ch

PL = direct Laspeyres. PP = direct Paasche,
L-ch = Laspeyres chain, P-ch = Paasche chain,

Now the Laspeyres-Paasche-Gap (LPG) is such that $P^L < P^P$, but chain indices are **again within the interval**

As now $P^L < P^P$ the covariance C between price and quantity relatives is **positive**: Problems with the "substitution bias"

$$\frac{P_{0t}^L}{P_{0t}^P} = 1 - \frac{C}{V_{0t}} < 1$$

Theorem of L. von Bortkiewicz

2.3.2 (1) Another modification: example with oscillating prices and quantities

variant of the example: **oscillations** (3 periods)

	ω_{1t}	ω_{2t}	λ_{1t}	λ_{2t}
1	1.2	0.9	2/3	0.75
2	0.75	5/3	5/3	1.2
3	10/9	2/3	0,9	10/9
4	1.2	0.9	2/3	0.75
5	0.75	5/3	5/3	1.2
6	10/9	8/15	0,9	10/9

links

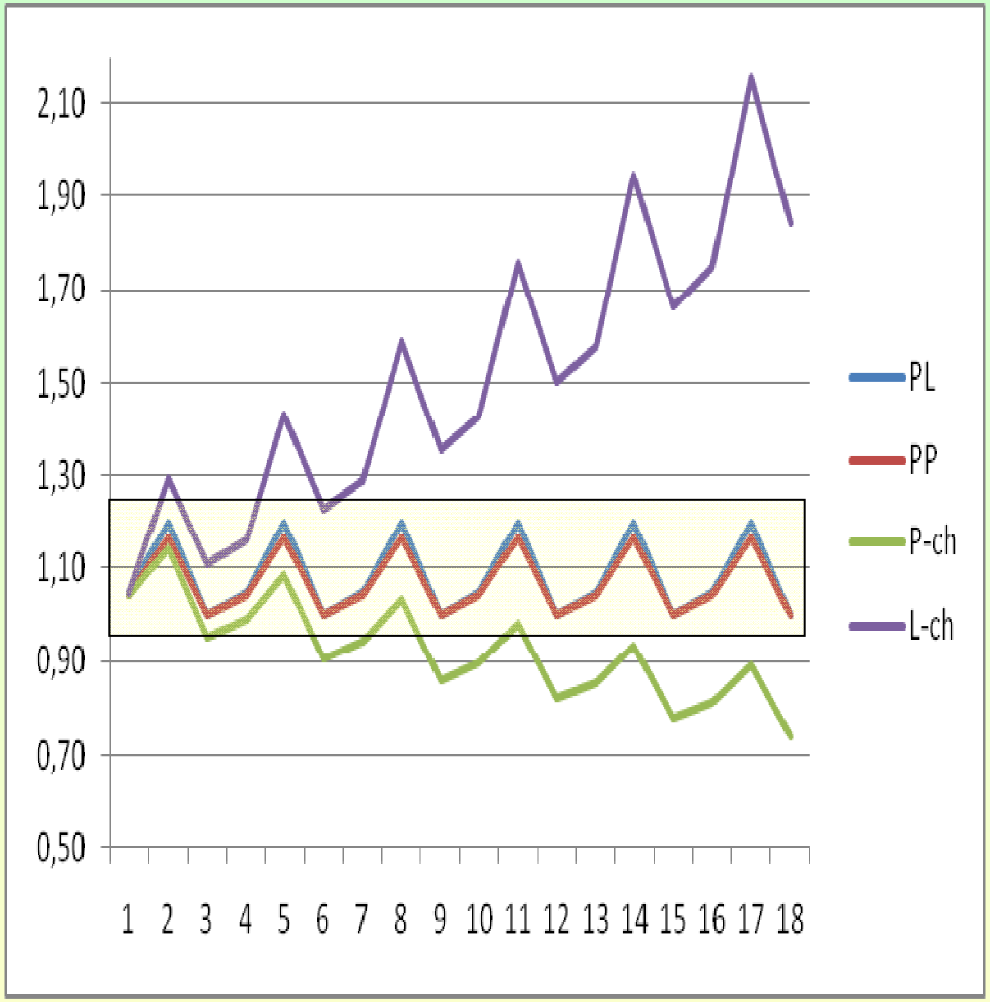
	P^P	Laspeyres	Paasche
1	1.0412	1,05	1,041176
2	1.1685	1,232268	1,096423
3	1	0.855792	0.833333
4	1.0412	1.05	1,041176
5	1.1685	1,232268	1,096423
6	1	0.855792	0.833333

	ω_{1t}	ω_{2t}	P^L
1	**1.2**	**0.9**	**1.05**
2	1.2*0.75=**0.9**	0.9*5/3=**1.5**	**1.2**
3	1.2*0.75*10/9=**1**	0.9*5/3*2/3=**1**	**1**
4	1.2	0.9	**1.05**
5	0.9	5/3	**1.2**
6	1	8/15	**1**

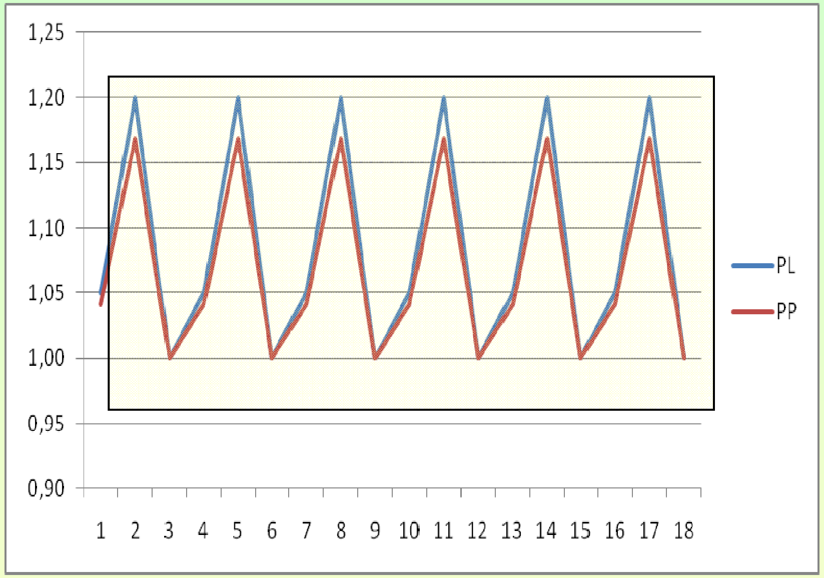
Direct indices P^L and P^P as well as the links (indices) are reflecting the cycle, but they have no trend. However, the **chain indices have also a trend**, up- or downwards:

Product of three links (over a cycle)
Laspeyres: 1.10724 (tend up)
Paasche: 0.95131 (trend down)

2.3.2 (2) Example with oscillating prices and quantities



Chain indices move **away from** the superlative direct **Fisher** index



arithmetic mean

$P^L = 1.08333$

$P^P = 1.06989$

$$\overline{P}^{L}_{0,17} = 2.1538 \qquad \overline{P}^{P}_{0,18} = 0.7412$$

note: all direct indices ≥ 1

2.3.3 (1) Some theoretical observations concerning the LPG/PLS (R. J. Hill)

Hill's theory of the LPS

1. deals with **two links only** $P_{12}P_{23,}$ and
2. primarily renders a **negative result**: →
3. results in relative **complicated** conditions in terms of **correlations**

Monotonic prices and quantities do not, in general, guarantee that chaining will reduce the PLS (= Paasche-Laspeyres-Spread)

Hill introduced three notions of monotonicity. The simplest reads as follows

$$\frac{p_{i2}}{p_{i1}} > 1 \Rightarrow \frac{p_{i3}}{p_{i2}} > 1 \qquad \frac{q_{i2}}{q_{i1}} > 1 \Rightarrow \frac{q_{i3}}{q_{i2}} > 1$$

He made a distinction between four correlation coefficients:

r-coefficient	prices	quantities	weights*
r_1	p_3/p_2	q_1/q_2	s_2
r_2	p_1/p_2	q_3/q_2	s_2
r_3	p_2/p_1	q_2/q_1	s_1
r_4	p_3/p_2	q_3/q_2	s_2

* expenditure shares

r_3, r_4 refer to the theorem of Bortkiewicz (see part II, sec. **5.2**)

Lemmas

(1) Laspeyres drift (1→3)
$D^{PL} < 1$ if and only if $r_1 > 0$

(2) Paasche drift $D_{PP} > 1$ iff $r_2 > 0$

Theorem
1.* sufficient to ensure $PLS^C < PLS^D$ is that r_1 and r_2 have the same sign which is opposite to r_3 and r_4.

* there are some other theorems

2.3.3 (2) Hill's theory of Laspeyres-Paasche Spread (and thus of approximating COLI)

1) Hill's theory does not seem to be easily generalized to more than just two links.

2) His empirical study of PLSC and PLSD of 22 3-period intervals (1-3, 2-4, 3-5, …, 20-22) and over the whole interval (1-22) of 22 weeks revealed

	PLS^D (direct)	PLS^C (chain)
3-period inter-vals (ranging from … to…)	1.0022 – 1.5415	1.0036 – 1.4900
Total interval (1-22)	1.0465	**2.5927** (!)*

In 7 out of 20 cases $PLS^C > PLS^D$ and only six 6 observations satisfied the sufficient conditions of theorem 1

3) Hill could find an example of $PLS^C > PLS^D$ although all notions of monotonicity applied, and an example for $PLS^C < PLS^D$ although monotonicity did not apply

Hence his most important finding was

Theorem 3: "Monotonicity … [three concepts] … are all neither necessary nor sufficient to ensure that chaining reduces the Paasche-Laspeyres spread"

Moreover: "Superlative (and most other) index number formulas tend to ***diverge*** from each other as the PLS rises"

* PLS^C "compounds" while PLS^D does not, which is – in Hill's view – due to the fact that there is a clear consumer (producer) substitution effect

3.1 List (overview) of **major shortcomings** of chain indices

A: Theory, interpretation and justification

- **A1**: no mean-value, no ratio of expenditures interpretation
- **A2**: chaining and constant update of weights inconsistent
- **A3**: most recent weights not always best weights
- **A4**: Divisia as a justification?

B: Axioms, **aggregation over time**

- **B1**: axioms apply to links only
- **B2**: non-linearity in prices p_t
- **B3**: path dependence
  - Determinants of drift → **3.5**
  - cyclical movement of prices

C: **Deflation** and **aggregation over sub-indices** → **part II**

- **C1**: no aggregative consistency
- **C2**: non additivity of volumes (SNA)
  - volumes not even proportional in the quantities

A problem that came up only recently

"Time consistency of **QNA – ANA**" → **part III**

→ If you like it easy you should be a non-chainer

3.2. Theoretical defects: overview

Theoretical considerations (dealing with the rationale of an approach) are possibly less compelling than the demonstration of unfavourable properties of an index function. It is, however of no small significance to

A1 give a verbal interpretation to a statistical figure

Does $\overline{P}_{0t}^{L} = 120$ mean that households have to spend 20% more in t than in 0 for …… ?

A2 examine whether or not the two main pur-poses of a chain index approach are reconcilable

chain indices are said to represent a device to make

- legitimate long term comparisons (transitivity), and the same time to make a
- constant up-date of weights

A3/A4 ask for a theo-retical justification of the two principal features of chain indices:

1. strive for most up-to-date weights, and
2. multiplying links

A theoretical foundation of chaining is (erroneous-ly) sometimes viewed in the Divisia index.

Irving Fisher for example was unable to find an index which was chainable **and** had variable weights (much later: proof of an inconsistency theorem (Funke 1979))

3.2.1 (1) Theoretical defects (group A): no traditional interpretations

None of the two "classical" interpretations of an index function applies

1) ratio of expenditures

$$\bar{P}_{0t}^{LC} = \frac{\sum p_t q_0^{LC}}{\sum p_0 q_0} \text{ where } q_{i,0}^{LC} = \frac{q_{i,t-1}}{Q_{0,t-1}^{PC}}$$

also true for direct Fisher

Not surprisingly: as the fixed-basket (with goods comparable over time) approach is abandoned there is no interpretation in terms of "expenditure" for a "basket" any more.

2) mean of relatives

$$P_{02}^{L} = \frac{p_{12}}{p_{10}} a + \frac{p_{22}}{P_{20}}(1-a) = m_1 a + m_2(1-a)$$

weights a and (1-a), p_{it} first subscript refers to good, second to period

$$\bar{P}_{02}^{L} = \left[\frac{p_{11}}{p_{10}} a + \frac{p_{21}}{p_{20}}(1-a)\right] \cdot \left[\frac{p_{12}}{p_{11}} b + \frac{p_{22}}{p_{21}}(1-b)\right]$$

$$\bar{P}_{02}^{L} = m_1 a[b + g(1-b)] + m_2(1-a)[(1-b) + b/g]$$

$$g = \frac{p_{11}p_{22}}{p_{12}p_{21}} = \frac{p_{22}/p_{21}}{p_{12}/p_{11}}$$

g=1 means same price change of both goods

Mean value property will be violated because sum of weights ≠ 1 →

3.2.1 (2) Chain index and mean-of-relatives formula

Obviously: if g = 1 in the case of two links

$$\overline{P}^L_{02} = m_1 a[b + g(1-b)] + m_2(1-a)[(1-b) + b/g] = m_1 a + m_2(1-a) = P^L_{02}$$

it can easily be seen that **equal changes of both prices in all periods** will yield $\overline{P}^L_{0t} = P^L_{0t}$

$g = g_1 = 1 \Rightarrow \frac{p_{12}}{p_{11}} = \frac{p_{22}}{p_{21}} = \lambda_1$ for three links assuming $g_1 = g_2 = 1 \Rightarrow \frac{p_{13}}{p_{12}} = \frac{p_{23}}{p_{22}} = \lambda_2$

gives $\overline{P}^L_{03} = m_1 a + m_2(1-a) = P^L_{03}$ and relatives $m_1 = \lambda_1\lambda_2 \frac{p_{11}}{p_{10}}$ $m_2 = \lambda_1\lambda_2 \frac{p_{21}}{p_{20}}$

However in $\overline{P}^L_{03} = \left[\frac{p_{11}}{p_{10}} a + \frac{p_{21}}{p_{20}}(1-a)\right] \cdot \left[\frac{p_{12}}{p_{11}} b + \frac{p_{22}}{p_{21}}(1-b)\right] \cdot \left[\frac{p_{13}}{p_{12}} c + \frac{p_{23}}{p_{22}}(1-c)\right]$

constant weights a = b = c will not necessarily result in $\overline{P}^L_{0t} = P^L_{0t}$

I originally thought there were no chain drift if weights were constant over time*

* However, it might have been better not to consider constant but price updated weights

t	weights	$p_{1t}/p_{1,t-1}$	$p_{2t}/p_{2,t-1}$	P^L_{0t}	$\overline{P}^L_{0t}$
1	a = 0.6	1.2	1.1	1.16	1.16
2	b = 0.6	1.3	1.2	1.464	1.4616
3	c = 0.6	1.05	1.18	1.60584	1.6106832

The mean value property is satisfied m_1 = 1.638 and m_2 = 1.5576

3.2.1 (3) Chain index and mean-of-relatives formula

t	weights	$p_{1t}/p_{1,t-1}$	$p_{2t}/p_{2,t-1}$	P^L_{0t}	$\overline{P}^L_{0t}$
1	a = 0.6	1.2	1.1	1.16	1.16
2	b = 0.6	1.3	1.2	1.464	1.4616
3	c = 0.6	1.05	1.18	1.60584	**1.61068**32

It is of course possible to find a weighted average (equal to 1.61068) of m_2 = 1.1*1.2*1.18=1.5576 and m_1 = 1.638

$$m_1 = \frac{p_{11}}{p_{10}}\frac{p_{12}}{p_{11}}\frac{p_{13}}{p_{12}} = \frac{p_{13}}{p_{10}} = 1.638$$

solving

$$\overline{P}^L_{03} = m_1 x + m_2(1-x) = 1.61068$$

for x gives **x = 0.66024** rather than **x = a = b = c = 0.6**

modification: assuming $m_1 = m_2$ (again constant weights)

t	weights	$p_{1t}/p_{1,t-1}$	$p_{2t}/p_{2,t-1}$	P^L_{0t}	$\overline{P}^L_{0t}$
1	a = 0.6	1.2	1.1	1.16	1.16
2	b = 0.6	1.3	1.2	1.464	1.4616
3	c = 0.6	1.1	1.3	**1.716**	1.724688

no mean value

gives

1.716 = $m_1$0.6+$m_2$0.4 and $m_1 = m_2$ = 1,2*1,3*1,1 = **1.716**

$\overline{P}^L_{0t}$ in different **sequences** (paths)

1.2	1,1	1,16
1.1	1.3	1.3688
1.3	1.2	**1.7247**

1.2	1.3	1.24
1.1	1.2	1.4136
1.3	1.1	1.724592

1.3	1.3	1,3
1.1	1.2	1.482
1.2	1.1	1.71912

Hence: also the sequence of identical price relatives matters!! (path dependence)

Tentative conclusion:
for the mean value property to be satisfied a small variance of price relatives appears more important than constancy of weights (expenditure shares)

The following numerical example demonstrates violation of the mean value property

	period 0		period 1		period 2	
i	prices	quantities	prices	quantities	prices	quantities
1	2	10	12	3	12	
2	5	4	7	10.29	14	

Price relatives 12/2 = 6 and 14/5 = 2.8

Direct Laspeyres: 4.4 (2.8 < 4.4 < 6)
Chain Laspeyres: 6.167 > 6

Representative result should be more important than **representative weights**

Weights in the formula $\overline{P}_{02}^{LC} = m_1 a\left[b + g(1-b)\right] + m_2 (1-a)\left[(1-b) + b/g\right]$

$m_1 a\left[b + g(1-b)\right]$: 0.833

$m_2 (1-a)\left[(1-b) + b/g\right]$: 0.4167

6***0.833** + 2.8***0.4167**

3.2.1 (5) Are violations of the mean value property relevant empirically?

Canadian Consumer Price Index (a chain index) March 1978

Goods	171.1
Services	171.4
Goods and Services	170.8

A similar problem (with the same cause, viz. multiplying links): non additivity of volumes

SNA §16.57: "A perverse form of non-additivity occurs when the chain index for the aggregate lies outside the range spanned by the chain indices for its components, a result that may be regarded as intuitively unacceptable by many users. This cannot be dismissed as very improbable. In fact it may easily occur when the range spanned by the components is very narrow and it has been observed on various occasions."

3.2.2 (1) Chainability and changing weights are inconsistent

Implicit assumptions of transitivity:

indices with different base period (weights) vary in proportion

$$P_{0t} = P_{0s} P_{st} \text{ it is implicitly assumed } \quad \frac{P_{0t}}{P_{0s}} = \frac{P_{st}}{P_{ss}}, \quad P_{ss} = 1$$

$$P_{0t} = P_{0r} P_{rs} P_{st} \quad \longrightarrow \quad \frac{P_{0t}}{P_{0r}} = \frac{P_{rt}}{P_{rr}} \text{ and } \frac{P_{rt}}{P_{rs}} = \frac{P_{st}}{P_{ss}}$$

Transitivity requires: indices with different base (weights) vary in proportion (**weights do not matter**)

On the other hand chaining is preferred because of adjustment of weights (**weights matter**): Are there transitive indices with variable weights?

This may be the reason for a well known inconsistency (dilemma) giving rise to a short historical note →

3.2.2 (2) Chainability and changing weights; historical note, Funke's theorem

Chain indices were destined to solve both problems, simultaneously

1) to arrive at consistent **long term** inter-temporal **comparisons** by multiplying over sub-intervals, **and** to
2) account for new situations by allowing for a **constant adjustment of weights**.

Aspect 2 has been one of the main reasons for **Alfred Marshall** to advocate chain indices. **Irving Fisher** already conjectured that you never get both "advantages" simultaneously: he saw there are chainable indices with constant weights and there are indices with variable weights violating chainability.

Theorem **Funke 1979**:*
The only index, satisfying the minimal requirements monotonicity, linear homogeneity, identity and commensurability and at the same time passing the circular test is the so called "**Cobb-Douglas index**" given by
(having constant weights α not related to expenditures)

$$P_{0t}^{CD} = \prod_{i=1}^{n} \left(\frac{p_{it}}{p_{i0}} \right)^{\alpha_i}$$

* "...that the main intention of the circular test, that is, the adjustment of the quantity weights to the new situation in each new dual comparison around a circle of periods or places cannot be accomplished. There **simply does not exist such a formula...**" (p. 685).

3.2.3 Most recent weights = best weights?

1. Operational definition and measure of (most) "relevant", "representative" ?

2. Assumptions needed to equate "last observed" and "most representative"

   There is no COLI justification of weights in terms of *needs and* an underlying *utility* functions. Two conditions may be stated, however

(1) the actually observed consumption structure is the **result of *voluntary* decisions** made by consumers, enjoying a real income by and large the same in 0 and in t, and

(2) the choice is ***not restricted***, and the variety among which a choice can be made is not altered by activities on the *supply* side.

(3) There should be at least some basis for verifying whether q_t *was chosen* because

> it was *preferred* to q_{t-1}, rather than because q_{t-1} was no longer available, or the *taste* (preferences) have changed; because COLI theory requires that the switch to q_t was made solely *in response* to changes in the *structure of prices* (on the basis of a given indifference of the representative consumer)

To sum up: most recent observed weights are not necessarily most "relevant" weights

Divisia in a nutshell:

Assume two (continuous in τ) functions, $p_i(\tau)$ and $q_i(\tau)$ exist for each commodity (i = 1, ... , n) at *any* point in time ($\tau \leq t$). By definition a value function $V(\tau)$ is given as follows

$$V(\tau) = \sum_{i=1}^{n} p_i(\tau) q_i(\tau) \quad \text{and} \quad V(\tau) = P(\tau)\,Q(\tau)$$

Unlike the function $V(\tau)$ the levels $P(\tau)$ and $Q(\tau)$ are unobservable.

They will lead eventually to a "price index" and "quantity index" respectively and are derived as follows

$$\frac{dV(\tau)/d\tau}{V(\tau)} = \frac{dP(\tau)/d\tau}{P(\tau)} + \frac{dQ(\tau)/d\tau}{Q(\tau)}$$

It is well known that the (continuous time) growth rate of a product is the sum of the growth rates of the factors.

The growth rate of P then is

$$\frac{dP(\tau)/d\tau}{P(\tau)} = \frac{d \ln P(\tau)}{d\tau} = \sum w_i(\tau) \frac{dp_i(\tau)/d\tau}{p_i(\tau)}$$

where $w_i(\tau) = p_i(\tau) q_i(\tau) / \sum p_i(\tau) q_i(\tau)$

growth rate of Q correspondingly

The essence: The logarithmic derivative (continuous time growth rate) of the unknown price level $P(\tau)$ is the weighted average of individual price levels $p_i(\tau)$ where weights $w_i(\tau)$ are expenditure shares at point τ (thus of course changing with time)

3.2.4 (2) Divisia index a justification for chain indices?

From growth rate to level: integration

integrals for P(t), Q(t), V(t)

$$P(t) = P(0)\exp\left(\int_0^t \sum w_i(\tau) \frac{d\ln p_i(\tau)}{d\tau} d\tau\right)$$

only V(t) is **not** path dependent

$$V_{0t} = \frac{V(t)}{V(0)} = \exp\left(\int_0^t \frac{dV(\tau)/d\tau}{V(\tau)} d\tau\right)$$

some properties of the "integral index" and chain indices are quite similar

Discrete time approximation

Diewert:

"The problem with this approach is that economic data are almost never available as continuous time variables ... Hence for empirical purpose it is necessary to approximate the continuous time Divisia price and quantity indexes by discrete time data. Since there are many ways of performing these approximations, the Divisia approach does not seem to lead to a definite result". (p. 23).

More important still, since the approximations "can differ *considerably* (in amount), the Divisia approach does not lead to a practical resolution of the price measurement problem" (p. 43).

3.3 (1) Poor axiomatic performance of chain indices

Axioms are functional equations (desirable properties) an index function should fulfil in order to be meaningful

example $f(\lambda \mathbf{p}, \mathbf{q}) = \lambda^r f(\mathbf{p},\mathbf{q})$ homogeneity of degree r in p
$f(\mathbf{p},\lambda \mathbf{q}) = f(\mathbf{p}, \mathbf{q})$ … of degree zero in quantities ($\mathbf{q}$)

More and less important axioms

More: derived from a concept of "prerequisites of measurement" and sensible analysis

concept of 100% and of a "unit of measurement":

identity, price dimensionality, commensurability

these are "invariance axioms"

correct reflection of direction and amount of change:

monotonicity, **linear homogeneity**

also analytically useful: good **aggregative** properties, product test

Less

Time reversal test $P_{t0} = 1/P_{0t}$
Factor reversal test (too restrictive)
Quantity reversal test
$P(\mathbf{p}_0, \mathbf{q}_t, \mathbf{p}_t, \mathbf{q}_0) = P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$

quantities of *both* periods must enter symmetrically the index formula (rules out Laspeyres depending on $\mathbf{q}_0$ only)

More about Irving **Fisher's kind of reasoning 3.3 (9)**

need for 1. *system* of axioms, and 2. *motivation* (and/or interpretation) of axioms

3.3 (2) More about some **fundamental** axioms (of binary price indices)

binary index → we not yet consider time series (aggregation over time)

in one stage → we not yet consider aggregation (over commodities) problems (multi-stage compilation)

then some elementary axioms appear as common-sense requirements of good measures:

- if **no change** takes place
  - then the index should not react P = 1
  - (1) **identity**
- if a **change** takes place, however, in a simple "stylized" fashion
  - an isolated change of **only one** price (ceteris paribus)
    - then the index should respond, change into the correct *direction*
    - (2) **monotonicity**
  - **all** prices change at the **same** rate λ
    - then P should be $P = \lambda$
    - (3) **proportionality**
  - prices change at **different rates** $\lambda_1, \lambda_2, \ldots, \lambda_n$
    - then the index should represent the average change
    - (4) **mean value property**

more specific: **additivity** → (2) monotonicity

identity is the special case $\lambda = 1$

These axioms are really **fundamental**, they are **all violated** in the case of chain indices

3.3 (3) Proportionality, mean value property and monotonicity

It is of little use to prefer the x-chain index to the y-chain index on axiomatic grounds

B1: Axioms apply to links only, not to the chain.
Links are indices, chains are not

Why are axioms so important? If an index function fails to properly reflect a simple (unrealistic) scenario it is unlikely that it will correctly mirror more complicated (realistic) situations.

We examine **mean value** property, **proportionality** and **monotonicity**

a direct Laspeyres index, or a Laspeyres **links satisfies all** these axioms yet a **chain** of two or more Laspeyres links will **violate them all**

Mean value property → this has been demonstrated already in sec. **3.2.1 (4)**

$$\min\left(\frac{p_{it}}{p_{i0}}\right) \leq P_{0t} \leq \max\left(\frac{p_{it}}{p_{i0}}\right)$$

Meaning:
A chain index may have **representative weights**, but P is not necessarily a **representative price-change** (i.e. price relative): **which sort of representativity seems to be more important?**

Proportionality $P(\mathbf{p}_0, \mathbf{q}_0, \lambda\mathbf{p}_0, \mathbf{q}_t) = \lambda$; **Identity**: $\lambda = 1$

the example shows that identitty may be violated (= axiom not satisfied)

	period 0		period 1		period 2	
i	prices	quantities	prices	quantities	prices	quantities
1	8	6	6	10	8	
2	12	4	15	5	12	

Direct Laspeyres: 1

Chain Laspeyres: 1*1.037 = 1.037

As to **identity** see also slide **1.2.2 (4)**

Why proportionality is violated?

$$P_1^{LC} P_2^{LC} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \frac{\sum \lambda p_0 q_1}{\sum p_1 q_1} \neq \lambda$$

as both price relatives are $\lambda_1 = \lambda_2 = 1$ also mean value property $\lambda_{min} \leq P \leq \lambda_{max}$ is violated

Monotonicity in prices p_t

$P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t^*, \mathbf{q}_t) > P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$ if $\mathbf{p}_t^* > \mathbf{p}_t$

$P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t^*, \mathbf{q}_t) < P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$ if $\mathbf{p}_t^* < \mathbf{p}_t$

strict monoton.

weak monotonicity $P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) \neq 1$ if $\mathbf{p}_t \neq \mathbf{p}_0$

	period 0		period 1		period 2	
i	prices	quantities	prices	quantities	prices	quantities
1	8	6	6	10	8	
2	12	4	15	5	11	

$\mathbf{p}_t = \mathbf{p}_2 \neq \mathbf{p}_0$

Direct Laspeyres: 92/96 = 0.9583

Chain Laspeyres: (96/96)*(135/135) = 1

Linearity in prices p_t

$P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t^*, \mathbf{q}_t) = P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) + P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t^{\Delta}, \mathbf{q}_t)$

if $\mathbf{p}_t^* = \mathbf{p}_t + \mathbf{p}_t^{\Delta}$ → more about **additivity** in chapter 6 (aggregation)

Non-linearity
in prices p_t

$$P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t^*, \mathbf{q}_t) \neq P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) + P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t^\Delta, \mathbf{q}_t)$$

assume $\mathbf{p}_t^* = \mathbf{p}_t + \mathbf{p}_t^\Delta$ in t = 1 and in t = 2 respectively

	period 0		period 1		period 2	
i	prices	quantities	prices	quantities	prices	quantities
1	2	10	5	3	10	
2	5	4	7	10	12	

Laspeyres price index P_{0t}

add vector p_t^Δ here (5, 3) or here (5, 3)

without vector

then the index P_{02} is given by

without vector	p_t^Δ added in period 1	p_t^Δ added in period 2
direct: **3.7**	direct: **3.7**	direct: **5.25** = **3.7 + 1.55**
chain: 1.95*1.76 = **3.44**	chain: 3.5*1.154 = **4.04**	chain: 1.95*2.29 = **4.47**

Index of $\mathbf{p}_t^\Delta$: **1.55**

The issue "non-linearity" will be resumed in **sec. 3.4.2**
The effect of p_t^Δ differs depending on **when** p_t^Δ is added.

3.3 (7) Linear homogeneity, relationships between some axioms

Linear Homogeneity
in prices p_t

An index function is said to be linear homogenous in prices p_t if

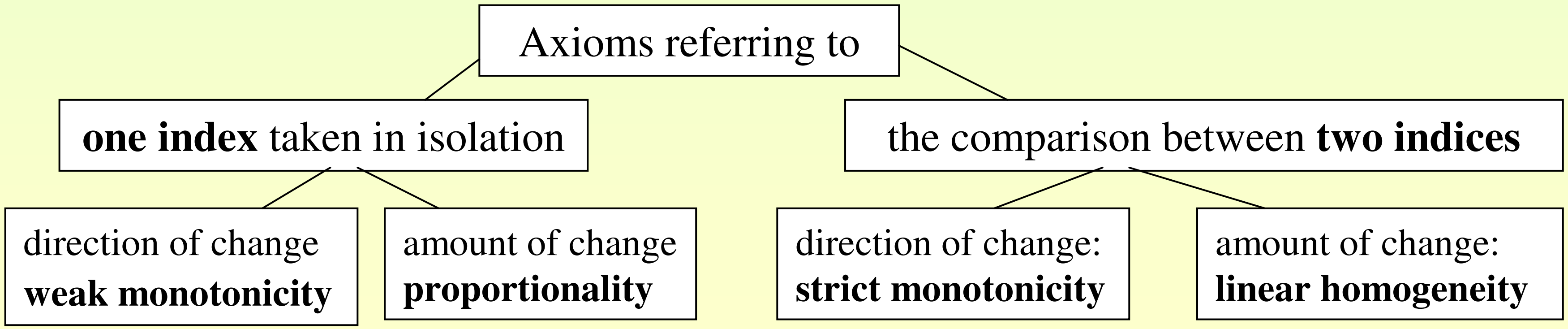
$$P(\mathbf{p}_0,\mathbf{q}_0,\lambda\mathbf{p}_t,\mathbf{q}_t) = \lambda P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t), \qquad \lambda \in \mathbb{R}$$

applied to three periods (two links) this could mean

$$(1)\ P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_1,\mathbf{q}_1,\lambda\mathbf{p}_2,\mathbf{q}_2) = \lambda P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_1,\mathbf{q}_1,\mathbf{p}_2,\mathbf{q}_2)$$

$$(2)\ P(\mathbf{p}_0,\mathbf{q}_0,\lambda\mathbf{p}_1,\mathbf{q}_1,\mathbf{p}_2,\mathbf{q}_2) = \lambda P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_1,\mathbf{q}_1,\mathbf{p}_2,\mathbf{q}_2)$$

which will hold (e.g. in the case of the Laspeyres chain index) because it simply amounts to replacing the link $\frac{\sum p_t q_{t-1}}{\sum p_{t-1} q_{t-1}}$ by the link $\frac{\sum \lambda p_t q_{t-1}}{\sum p_{t-1} q_{t-1}} = \lambda \frac{\sum p_t q_{t-1}}{\sum p_{t-1} q_{t-1}}$



3.3 (8) Proportionality and linear homogeneity

if linear homogeneity and identity are met → then also (strict) proportionality

the converse is (as usual) not true

not fulfilled (e.g. in the case of chain indices) [refers to "identity"]

not fulfilled (e.g. in the case of chain indices) [refers to "proportionality"]

	linear homogeneity	no linear homogeneity
proportionality	**direct** Paasche, Laspeyres, Fisher	**direct**: Vartia I indices of G. Stuvel Exponential index
no proportionality	**chain indices (all)** value index	Drobisch's unit value index (direct index)

$$P_{0t}^{EX} = \ln\left[\omega_i \sum \exp\left(\frac{p_{it}}{p_{i0}}\right)\right]$$

One may, however, have widely different views regarding the relevance of an axiom ⇒ **Fisher** and the mere number of axioms fulfilled cannot be a criterion

the table demonstrates independence of axioms

axiomatic record

one axiom satisfied: linear homogeneity

four axioms violated: proportionality (identity), monotonicity, mean value property, additivity

3.3 (9) The legacy of Irving Fisher: a fundamentally different axiomatic approach

Irving Fisher introduced (or at least emphasized very much)

- **reversal** tests (Commodity, time, product) and
- **crossing** (= averaging) of weights and formulas → $P_{0t}^{F} = \sqrt{P_{0t}^{L} P_{0t}^{P}}$

for which he liked to give a justification in terms of "fairness" and "symmetry"; for him formulas like P^L and P^P were equally well justified and therefore he took a (geometric) average. He also made "double crossing" (i.e. crossing of crossed formulas), a way by which he arrived at the formula

$$\frac{\sum p_t q_t}{\sum p_0 q_0}\left[\frac{\sum p_0 q_0 \frac{q_0}{q_t} \sum p_t q_0 \left(\frac{q_0}{q_t} + \frac{p_t}{p_0}\right) \sum p_t q_t \frac{p_t}{p_0} \left(\sum p_0 q_t\right)^2}{\sum p_0 q_0 \sum p_t q_0 \sum p_t q_t \frac{q_t}{q_0} \sum p_0 q_t \left(\frac{q_t}{q_0} + \frac{p_0}{p_t}\right)}\right]^{1/4}$$

For Fisher this index had a better "test-record" than his own "ideal index" (he focussed on purely formal aspects)

In his days some means like logarithmic, exponential , or power mean were not yet known. Also deflation in the framework of National Accounts was not yet an issue, and aggregation properties were not yet found relevant. Fisher's thinking lives on in many countries (esp. in the USA) but differs fundamentally from the (former) German index-tradition.

3.4 A short look at aggregation properties (overview) *more in part II*

aggregation over

commodities (sub-aggregates)

time (temporal aggregation)

Our next steps

group B2 and B3 of our arguments against chain indices

1. how and **total index** (of prices, or quantities) is related to the **partial indices** (of individual goods)
2. the same concerning absolute figures (e.g. **volumes**) **gained by** using **chain indices** (e.g. as deflators)

   (more about this in part II)

3. analysis of how the increase or decrease of the chain price index is related to the increase/decrease of prices of the individual goods (consequences of **non-linearity in the prices** $\mathbf{p_t}$)
4. **path dependence**, cyclical movement of prices

Paul **Schreyer** in a rebuff of the argument of non-additivity against chaining: "But analytical arguments not always convincing – it is not always clear for which analytical purpose constant price-levels are really needed" (slide 18)

3.4.1 (1) Aggregation over commodities: direct Laspeyres quantity index:

The **simple** situation of the **direct Laspeyres index**

Sub-aggregates A and B are added to the total aggregate S (analogous formulas for price indices)

$$Q_{01}^{L(S)} = \frac{\sum q_{i1}^{A} p_{i1}^{A} + \sum q_{i1}^{B} p_{i1}^{B}}{\sum q_{i0}^{A} p_{i0}^{A} + \sum q_{i0}^{B} p_{i0}^{B}}$$

$$Q_{01}^{L(S)} = Q_{01}^{L(A)} w_0^A + Q_{01}^{L(B)} w_0^B = \frac{\sum q_{i1}^{A} p_{i0}^{A}}{\sum q_{i0}^{A} p_{i0}^{A}} w_0^A + \frac{\sum q_{i1}^{B} p_{i0}^{B}}{\sum q_{i0}^{B} p_{i0}^{B}} w_0^B$$

constant weights $w_0^A = \frac{\sum q_{i0}^{A} p_{i0}^{A}}{\sum q_{i0}^{A} p_{i0}^{A} + \sum q_{i0}^{B} p_{i0}^{B}}$ and $w_0^B = \frac{\sum q_{i0}^{B} p_{i0}^{B}}{\sum q_{i0}^{A} p_{i0}^{A} + \sum q_{i0}^{B} p_{i0}^{B}}$

$$Q_{02}^{L(S)} = Q_{02}^{L(A)} w_0^A + Q_{02}^{L(B)} w_0^B = \frac{\sum q_{i2}^{A} p_{i0}^{A}}{\sum q_{i0}^{A} p_{i0}^{A}} w_0^A + \frac{\sum q_{i2}^{B} p_{i0}^{B}}{\sum q_{i0}^{B} p_{i0}^{B}} w_0^B$$

$$Q_{03}^{L(S)} = Q_{03}^{L(A)} w_0^A + Q_{03}^{L(B)} w_0^B = \frac{\sum q_{i3}^{A} p_{i0}^{A}}{\sum q_{i0}^{A} p_{i0}^{A}} w_0^A + \frac{\sum q_{i3}^{B} p_{i0}^{B}}{\sum q_{i0}^{B} p_{i0}^{B}} w_0^B$$

3.4.1 (2) Aggregation over commodities: chain Laspeyres quantity index

The much more **complicated** situation of the **chain Laspeyres index**

Sub-aggregates A and B are added to the total aggregate S (analogous formulas for price indices)

aggregate chain-index

$$\overline{Q}_{02}^{L(S)} = Q_{01}^{L(S)}\left[Q_{12}^{L(A)} w_1^A + Q_{12}^{L(B)} w_1^B\right] = Q_{01}^{L(S)}\left[\frac{\sum q_{i2}^A p_{i1}^A}{\sum q_{i1}^A p_{i1}^A} w_1^A + \frac{\sum q_{i2}^B p_{i1}^B}{\sum q_{i1}^B p_{i1}^B} w_1^B\right]$$

$$\overline{Q}_{03}^{L(S)} = \overline{Q}_{02}^{L(S)}\left[Q_{23}^{L(A)} w_2^A + Q_{23}^{L(B)} w_2^B\right] = Q_{02}^{L(S)}\left[\frac{\sum q_{i3}^A p_{i2}^A}{\sum q_{i2}^A p_{i2}^A} w_2^A + \frac{\sum q_{i3}^B p_{i2}^B}{\sum q_{i2}^B p_{i2}^B} w_2^B\right]$$

variable weights

$$w_0^A = \frac{\sum q_{i0}^A p_{i0}^A}{\sum q_{i0}^A p_{i0}^A + \sum q_{i0}^B p_{i0}^B} \qquad w_1^A = \frac{\sum q_{i1}^A p_{i1}^A}{\sum q_{i1}^A p_{i1}^A + \sum q_{i1}^B p_{i1}^B} \qquad w_2^A = \frac{\sum q_{i2}^A p_{i2}^A}{\sum q_{i2}^A p_{i2}^A + \sum q_{i2}^B p_{i2}^B}$$

$$w_t^B = \frac{\sum q_{it}^B p_{it}^B}{\sum q_{it}^A p_{it}^A + \sum q_{it}^B p_{it}^B}$$

sectoral chain-indices

$$\overline{Q}_{03}^{L(A)} = Q_{01}^{L(A)} Q_{12}^{L(A)} Q_{23}^{L(A)} = \frac{\sum q_{i1}^A p_{i0}^A}{\sum q_{i0}^A p_{i0}^A} \cdot \frac{\sum q_{i2}^A p_{i1}^A}{\sum q_{i1}^A p_{i1}^A} \cdot \frac{\sum q_{i3}^A p_{i2}^A}{\sum q_{i2}^A p_{i2}^A}$$

$$\overline{Q}_{03}^{L(B)} = Q_{01}^{L(B)} Q_{12}^{L(B)} Q_{23}^{L(B)} = \frac{\sum q_{i1}^B p_{i0}^B}{\sum q_{i0}^B p_{i0}^B} \cdot \frac{\sum q_{i2}^B p_{i1}^B}{\sum q_{i1}^B p_{i1}^B} \cdot \frac{\sum q_{i3}^B p_{i2}^B}{\sum q_{i2}^B p_{i2}^B}$$

remember slide 1.1.3 (2) aggregation "unproblematisch"

3.4.1 (3) Aggregation of chain Laspeyres quantity index: numerical **example 1**

	A1		A2		B1		B2	
t	p	q	p	q	p	q	p	q
0	3	3	10	5	4	2	9	6
1	6	6	12	6	12	5	11	7
2	7	5	15	7	18	4	14	8
3	9	8	18	9	25	7	17	10

Example 1

similar quantity (and price) movement in both sectors, A and B thus **almost constant weights** w^A and $w^B = 1 - w^A$

Both, prices and quantities are going up in both sectors, A and B; next example: $q_A \uparrow$ and $q_B \downarrow$

t	$\overline{Q}_{0t}^{L(A)}$	$\overline{Q}_{0t}^{L(B)}$	$\overline{Q}_{0t}^{L(S)}$	w_t^A	$Q_{0t}^{L(A)}$	$Q_{0t}^{L(B)}$	$Q_{0t}^{L(S)}$
0				0.4876			
1	132.20	133.87	133.06	0.4408	132.20	133.87	133.06
2	139.58	132.89	135.77	0.4321	144.07	141.94	130.58
3	190.38	192.11	191.15		193.22	190.32	191.74

mean value fulfilled 190 < 191 < 192

here constant weights $w^A = 0.4876$

3.4.1 (4) Aggregation of chain Laspeyres quantity index: numerical **example 2**

t	A1		A2		B1		B2	
t	p	q	p	q	p	q	p	q
0	6	40	10	16	4	80	9	60
1	9	60	12	22	12	65	11	52
2	12	180	15	37	18	45	14	48
3	15	220	18	120	25	38	17	40

Example 2

different quantity (and price) movement in the sectors, A and B $q_A \uparrow$ and $q_B \downarrow$, thus **constantly changing weights** w^A and $w^B = 1 - w^A$

t	$\overline{Q}_{0t}^{L(A)}$	$\overline{Q}_{0t}^{L(B)}$	$\overline{Q}_{0t}^{L(S)}$	$Q_{0t}^{L(A)}$	$Q_{0t}^{L(B)}$	$Q_{0t}^{L(S)}$	w_t^A
0							0.6825
1	145.00	84.65	103.81	145.00	84.65	103.81	0.6271
2	372.24	66.87	193.29	362.50	71.16	163.65	0.3531
3	608.74	56.13	261.77	630.00	59.53	240.63	0.2299

weights are changing dramatically (0.68 → 0.23) and constant weights appear unrealistic

3.4.1 (5) Aggregation of quantity indices: estimation of absolute volumes

Example 1

t	$\overline{Q}_{0t}^{L(A)}$	$\overline{Q}_{0t}^{L(B)}$	sum	$\overline{Q}_{0t}^{L(S)}$	$Q_{0t}^{L(A)}$	$Q_{0t}^{L(B)}$	sum	$Q_{0t}^{L(S)}$
	(1)	(2)		(1)+(2)	(3)	(4)		(3)+(4)
0	59 [(a)]	62 [(a)]	121	121	59	62	121	121
1	78 [(b)]	83	161	161	78	83	161	161
2	82.35 [(c)]	82.39 [(c)]	164.74	164.28	85 [(c)]	88	158	158
3	112.32	119.11	231.43	231.29	114	118	232	232

The entries in this table are related to slide **3.4.1 (3)** as follows

(a) 59 = 3*3+10*5 ($\Sigma p_0 q_0$ for aggregate A), correspondingly 62 = 4*2+9*6 (for B)

(b) 59*1.322 = 77.998 ≈ 78

(c) 82.35 = 59*1.3958, likewise
82.39 = 62*1.3289, and 85 = 59*1.4407

In the same way the figures in the following table of example 2 are related to the figures in **3.4.1 (4)**

Additivity is only slightly violated (231.43 ≈ 231.29), also 231 is not much different from 232. However, this will change in the following example 2.

3.4.1 (6) Aggregation of quantity indices: estimation of absolute volumes

Example 1

t	$\overline{Q}_{0t}^{L(A)}$	$\overline{Q}_{0t}^{L(B)}$	sum	$\overline{Q}_{0t}^{L(S)}$	$Q_{0t}^{L(A)}$	$Q_{0t}^{L(B)}$	sum	$Q_{0t}^{L(S)}$
	(1)	(2)		(1)+(2)	(3)	(4)		(3)+(4)
3	112.32	119.11	231.43	231.29	114	118	232	232

Example 2

	(1)	(2)	sum	(1)+(2)	(3)	(4)	sum	(3)+(4)
1	exactly the same like direct index				580 (a)	728 (b)	1308	1308
2	1488.96	575.08	2435.45 (c)	2064.04	1450 (d)	612	2062	2062
3	2434.96	482.72	3298.30 (e)	2917.68	2520	512	3032	3032

Non-additivity is now more pronounced than in example 1

(a) 580 = (6*40+10*16)*1.45 = 400*1.45, **(b)** 728 = (4*80+9*60)*0.8465 = 860*0.8465
(c) 2435.45 = (400+860)*1.9329, **(d)** 1450 = 400*3.625, **(e)** = (400+860)*2.6177

3.4.2 (1) Difference between indices: Nonlinearity in prices of t (= shortcoming B2)

given absolute increases of prices Δp_1, Δp_2, …

1. direct Laspeyres

$$P_{02}^{L} = 1 + \frac{\sum q_0 \Delta p_1}{\sum q_0 p_0} + \frac{\sum q_0 \Delta p_2}{\sum q_0 p_0}$$

$$P_{03}^{L} = P_{02}^{L} + \frac{\sum q_0 \Delta p_3}{\sum q_0 p_0}$$

differences can be accounted for to individual price differences Δp_i

2. Laspeyres chain index

$$\overline{P}_{02}^{L} = \left(\frac{\sum q_0 \Delta p_1}{\sum q_0 p_0} + 1 \right) \left(\frac{\sum q_1 \Delta p_2}{\sum q_1 p_1} + 1 \right)$$

Now a (more complicated) multiplication between successive indices takes place

$$= \left(1 + \frac{\sum q_0 \Delta p_1}{\sum q_0 p_0} \right) + \left[\frac{\sum q_1 \Delta p_2}{\sum q_1 p_1} \left(1 + \frac{\sum q_0 \Delta p_1}{\sum q_0 p_1} \right) \right]$$

$$\overline{P}_{03}^{L} = \overline{P}_{02}^{L} \left(\frac{\sum q_2 \Delta p_3}{\sum q_2 p_2} + 1 \right) = \overline{P}_{02}^{L} + \left[\sum \Delta p_3 \frac{q_2}{\sum q_2 p_2} \overline{P}_{02}^{LC} \right]$$

3.4.2 (2) Nonlinearity in prices of t (B2): direct index is linear

Linearity
in prices p_t

$$P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t^*, \mathbf{q}_t) = P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) + P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t^{\Delta}, \mathbf{q}_t)$$

if $\mathbf{p}_t^* = \mathbf{p}_t + \mathbf{p}_t^{\Delta}$ in this case

$$P(\mathbf{p}_0, \mathbf{p}_1) = P(\mathbf{p}_0, \mathbf{q}_0) + P(\mathbf{p}_0, \Delta\mathbf{p}_1), \quad \ldots$$

$$P(\mathbf{p}_0, \mathbf{p}_t) = P(\mathbf{p}_0, \mathbf{q}_{t-1}) + P(\mathbf{p}_0, \Delta\mathbf{p}_t)$$

	period 0		period 1		period 2	
i	prices	quantities	prices	quantities	prices	quantities
1	8	10	10	9	12	
2	12	4	13	5	14	

$$\Delta\mathbf{p}_1 = \Delta\mathbf{p}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ 13 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 12 \\ 14 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Equal changes in prices = equal effects

$$P_{01}^{L} = \frac{128}{128} + \frac{24}{128} = \frac{152}{128} = 1.1875$$ direct Laspeyres

as opposed to chain Laspeyres $$P_{02}^{L} = \frac{152}{128} + \frac{24}{128} = \frac{176}{128} = 1.375$$

3.4.2 (3) Nonlinearity in prices of t (B2): chain index is nonlinear

$$\overline{P}_{02}^{L} = \left(\frac{\sum q_0 \Delta p_1}{\sum q_0 p_0} + 1\right)\left(\frac{\sum q_1 \Delta p_2}{\sum q_1 p_1} + 1\right)$$

Now: Equal changes in prices = unequal effects*

	period 0		period 1		period 2	
i	prices	quantities	prices	quantities	prices	quantities
1	8	10	10	9	12	
2	12	4	13	5	14	

chain: $$\overline{P}_{02}^{L} = \left(1 + \frac{28}{128}\right) \cdot \left(1 + \frac{23}{155}\right) = 1.1875 \cdot 1.1487 = 1.3637$$

direct: $$P_{02}^{L} = 1 + \frac{28}{128} + \frac{28}{128} = \frac{176}{128} = 1.375$$

* This would apply also if quantities q_1 were equal to q_0, because $1.1875 * 1.1875 = 1.4102 \neq 1.375$

3.5 Path dependence and the determinants of the drift

3.5.1 introduces the **drift functions** in terms of growth rates and temporal covariances and examines the relationships between them (1 - 4).
The formulas are verified showing how chain drift is determined by the covariance **(5 slides 86 – 90)**

3.5.2 an example with "bouncing" prices is worked out over **five cycles** showing the consequences for direct as well as chained indices of both, prices as well as quantities **(5 slides 91 – 95)**

An example of a 2-period-cycle is given in v.d.Lippe 2001, ch.3.4.b
it is also included in the annex of the formula handouts for this course

3.5.3 shows how the **drift functions** and the **Laspeyres Paasche Gap** (LPG) are related, and how the LPG between **chained** Laspeyres and Paasche price index develop, making use of a theorem of Ladislaus von Bortkiewicz* **(3 slides 96 – 98)**

* the theorem itself will be presented in section 5

1. The general idea of **path-dependence** (no transitivity) has already been described in sec. **1.2.2 (4)**

$$\overline{P}_{0t}^{L} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \frac{\sum p_2 q_1}{\sum p_1 q_1} \cdots \frac{\sum p_t q_{t-1}}{\sum p_{t-1} q_{t-1}} \neq P_{0t}^{L} = \frac{\sum p_t q_0}{\sum p_0 q_0}$$

t = 0		t = 1		t = 2		t = 3		t = 4	
p	q	p	q	p	q	p	q	p	q
2	10	4	12	3	20	1	16	2	10
5	20	3	15	4	10	4	12	5	20

It is precisely this example that will be worked out for more than just one cycle ⇒

Two meanings of path dependence:

1. Direct ($P^L = 1$) ≠ chain index, in particular "chain drift" = no multi-period identity
2. Chain index depends on how interval is partitioned
   annually **0.742**
   biannually **0.825**

2. The purpose of the **drift function** is to measure the deviation from transitivity. It is a function of the interval in question (0, t) [**note $D_{01} = 1$**] and the kind of index (e.g. Laspeyres price index, or Paasche quantity index)

$$D_{0t}^{PL} = \overline{P}_{0t}^{L} / P_{0t}^{L}$$

$$D_{0t}^{QP} = \overline{Q}_{0t}^{P} / Q_{0t}^{P}$$

3.5.1 (2) Path dependence and drift function: definitions of the drift function

Drift function (e.g. of Laspeyres price index) is **recursive** and can be **expressed in** terms of

see already slide 20

growth rates

$$g_t^k = \frac{\sum p_t q_k}{\sum p_{t-1} q_k} = \sum \frac{p_t}{p_{t-1}} \frac{p_{t-1} q_k}{\sum p_{t-1} q_k}$$

$$D_{02}^{PL} = \frac{g_1^0 g_2^1}{g_1^0 g_2^0} = \frac{\overline{P}_{02}^{LC}}{P_{02}^L} = \frac{P_{01}^L P_{12}^L}{P_{02}^L} = \frac{P_{12}^L}{P_{02}^L / P_{01}^L} = \frac{g_2^1}{g_2^0}$$

$$D_{02}^{PL} = \frac{g_2^1}{g_2^0}$$

The recursive systems shows how drift changes with the passage of time

$$D_{03}^{PL} = \frac{g_2^1 g_3^2}{g_2^0 g_3^0} = D_{02}^{PL} \frac{g_3^2}{g_3^0}$$

intertemporal **correlations**

$$D_{02}^{PL} = \frac{\mathrm{Cov}(x_{12}, y_{01})}{\bar{x}_{12} \cdot \bar{y}_{01}} + 1$$

$$D_{03}^{PL} = D_{02}^{PL} \left(\frac{\mathrm{Cov}(x_{23}, y_{02})}{\bar{x}_{23} \cdot \bar{y}_{02}} + 1 \right) \text{etc.}$$

$$x_{i,12} = \frac{p_{i2}}{p_{i1}},\ x_{i,23} = \frac{p_{i3}}{p_{i2}}, \ldots \text{(links)}$$

$$y_{i,01} = \frac{q_{i1}}{q_{i0}},\ y_{i,02} = \frac{q_{i2}}{q_{i0}}, \ldots \text{(relatives)}$$

weights in the covariances

$$w_{y:01}^{x:12} = p_1 q_0 \Big/ \sum p_1 q_0 \qquad w_{y:02}^{x:23} = p_2 q_0 \Big/ \sum p_2 q_0$$

3.5.1 (3) Relation between the definitions of the drift function

t-1	$\bar{x}_{t-1,t}$	$\bar{y}_{0,t-1} = Q^P_{0,t-1}$	$Cov(x_{t-1,t}, y_{0,t-1}) / Q^P_{0.t-1}$
1	$\frac{\sum p_2 q_0}{\sum p_1 q_0} = g_2^0$	$\frac{\sum p_1 q_1}{\sum p_1 q_0} = Q^P_{01}$	$\frac{\sum p_2 q_1}{\sum p_1 q_1} - \frac{\sum p_2 q_0}{\sum p_1 q_0} = g_2^1 - g_2^0$
2	$\frac{\sum p_3 q_0}{\sum p_2 q_0} = g_3^0$	$\frac{\sum p_2 q_2}{\sum p_2 q_0} = Q^P_{02}$	$\frac{\sum p_3 q_2}{\sum p_2 q_2} - \frac{\sum p_3 q_0}{\sum p_2 q_0} = g_3^2 - g_3^0$
3	$\frac{\sum p_4 q_0}{\sum p_3 q_0} = g_4^0$	$\frac{\sum p_3 q_3}{\sum p_3 q_0} = Q^P_{03}$	$\frac{\sum p_4 q_3}{\sum p_3 q_3} - \frac{\sum p_4 q_0}{\sum p_3 q_0} = g_4^3 - g_4^0$

$$D^{PL}_{02} = \left(\frac{Cov(x_{12}, y_{01})}{\bar{x}_{12} \cdot \bar{y}_{01}} + 1 \right) = \left(\frac{Cov(x_{12}, y_{01})}{\bar{x}_{12} \cdot Q^P_{01}} + 1 \right) = \frac{g_2^1 - g_2^0}{g_2^0} + 1 = \frac{g_2^1}{g_2^0}$$

$$D^{PL}_{03} = D^{PL}_{02} \left(\frac{Cov(x_{23}, y_{02})}{\bar{x}_{23} \cdot \bar{y}_{02}} + 1 \right) = \frac{g_2^1}{g_2^0} \frac{g_3^2}{g_3^0}$$

$$D^{PL}_{04} = D^{PL}_{02} D^{PL}_{03} \left(\frac{Cov(x_{34}, y_{03})}{\bar{x}_{32} \cdot \bar{y}_{03}} + 1 \right) = \frac{g_2^1}{g_2^0} \frac{g_3^2}{g_3^0} \frac{g_4^3}{g_4^0}$$

For the simple reason that:

$$P^L_{03} = g_1^0 g_2^0 g_3^0$$

and

$$\bar{P}^{LC}_{03} = g_1^0 g_2^1 g_3^2$$

3.5.1 (4) Drift function and violation of identity

$$D_{02}^{PL} = \frac{Cov(x_{12}, y_{01})}{\bar{x}_{12} \cdot \bar{y}_{01}} + 1 = \frac{g_2^1}{g_2^0}$$

$$Cov(x_{23}, y_{02}) = Q_{01}^{P}(g_2^1 - g_2^0)$$

example: violation of identity (slide 69) the chain Laspeyres index was **1.037 > 1**

	period 0		period 1		period 2	
i	prices	quantit-ies	prices	quantit-ies	prices	quanti-ties
1	8	6	6	10	8	
2	12	4	15	5	12	

$Q_{01}^{P} = 135/96 = 1.406$

$g_2^0 = 1$

$D_{02}^{PL} = \bar{P}_{02}^{L} = g_2^1 = 1.037$

$= 140 \ /135$

$Cov(p_2/p_1, q_1/q_0)$

$= +0.05208$

modification (again $g_2^0 = 1$)

	period 0		period 1		period 2	
i	prices	quanti-ties	prices	quanti-ties	prices	quanti-ties
1	8	6	6	5	8	
2	12	4	15	10	12	

$D_{02}^{PL} = \bar{P}_{02}^{L} = g_2^1 = 0.889$

$= 160 \ /180$

$Q_{01}^{P} = 180/96 = 1.875$

$Cov(p_2/p_1, q_1/q_0)$

$= -0.20833$

Chain indices quite obviously do **not** provide **a *pure* price comparison** (only quantities q_1 differ)

3.5.1 (5) Verifying the relationships: example with cyclical price-movement

The general rule: **if** $\mathrm{Cov}(x_{t-1,t}, y_{0,t-1}) > 0$ **then** increasing drift $D_{0t}^{PL} > D_{0,t-1}^{PL}$

if $\mathrm{Cov}(x_{t-1,t}, y_{0,t-1}) < 0$ **then** decreasing drift $D_{0t}^{PL} < D_{0,t-1}^{PL}$

t = 0		t = 1		t = 2		t = 3		t = 4	
p	q	p	q	p	q	p	q	p	q
2	10	4	12	3	20	1	16	2	10
5	20	3	15	4	10	4	12	5	20

in terms of the covariance

$$D_{02}^{PL} = \frac{\mathrm{Cov}(x_{12}, y_{01})}{\bar{x}_{12} \cdot \bar{y}_{01}} + 1$$

$$= \frac{g_2^1 - g_2^0}{\bar{x}_{12}} + 1 = \frac{g_2^1 - g_2^0}{g_2^0} + 1$$

$$= \frac{96/93 - 1.1}{1.1} + 1 = \frac{-0.0677}{1.1} + 1 = 0.9384$$

in terms of growth factors

$$g_1^0 = \frac{4 \cdot 10 + 3 \cdot 20}{2 \cdot 10 + 5 \cdot 20} = \frac{100}{120} = 0.833$$

$$D_{02}^{PL} = \frac{g_2^1}{g_2^0} = \frac{96/93}{110/100} = \frac{1.0323}{1.1} = 0.9384$$

$$D_{03}^{PL} = \frac{g_2^1 g_3^2}{g_2^0 g_3^0} = 0.9384 \cdot \frac{0.6}{0.8181} = 0.6882$$

$$D_{03}^{PL} = D_{02}^{PL}\left(\frac{\mathrm{Cov}(x_{23}, y_{02})}{\bar{x}_{23} \cdot Q_{02}^{P}} + 1\right)$$

$$D_{03}^{PL} = D_{02}^{PL}\left(\frac{g_3^2 - g_3^0}{\bar{x}_{23}} + 1\right) = D_{02}^{PL} \cdot \frac{g_3^2}{g_3^0}$$

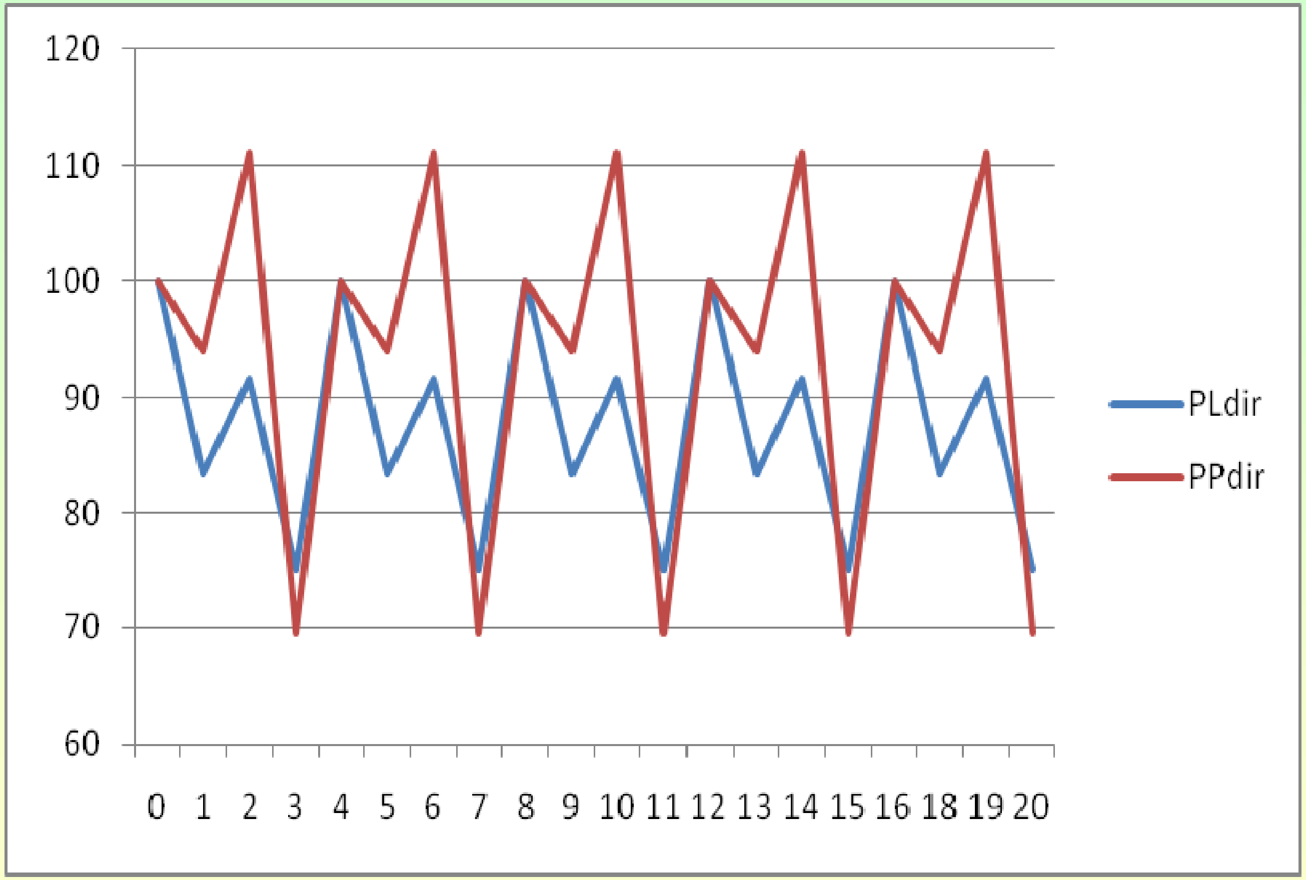
The example now will be continued assuming 5 cycles of a length of four periods (0-3), (4-7),…

3.5.2 (1) Example with 5 cycles (Cyclical movement of prices, "bouncing")

	pa	pb	qa	qb
0	2	5	10	20
1	4	3	12	15
2	3	4	20	10
3	1	4	16	12
4	2	5	19	20
5	4	3	12	15
6	3	4	20	10
7	1	4	16	12
8	2	5	19	20
9	4	3	12	15
10	3	4	20	10
11	1	4	16	12
12	2	5	19	20
13	4	3	12	15
14	3	4	20	10
15	1	4	16	12
16	2	5	19	20
18	4	3	12	15
19	3	4	20	10
20	1	4	16	12

the same example as above (slide 86)

Oscillating prices and quantities (example of 1.2.2 (4) – four periods cycle – continued)

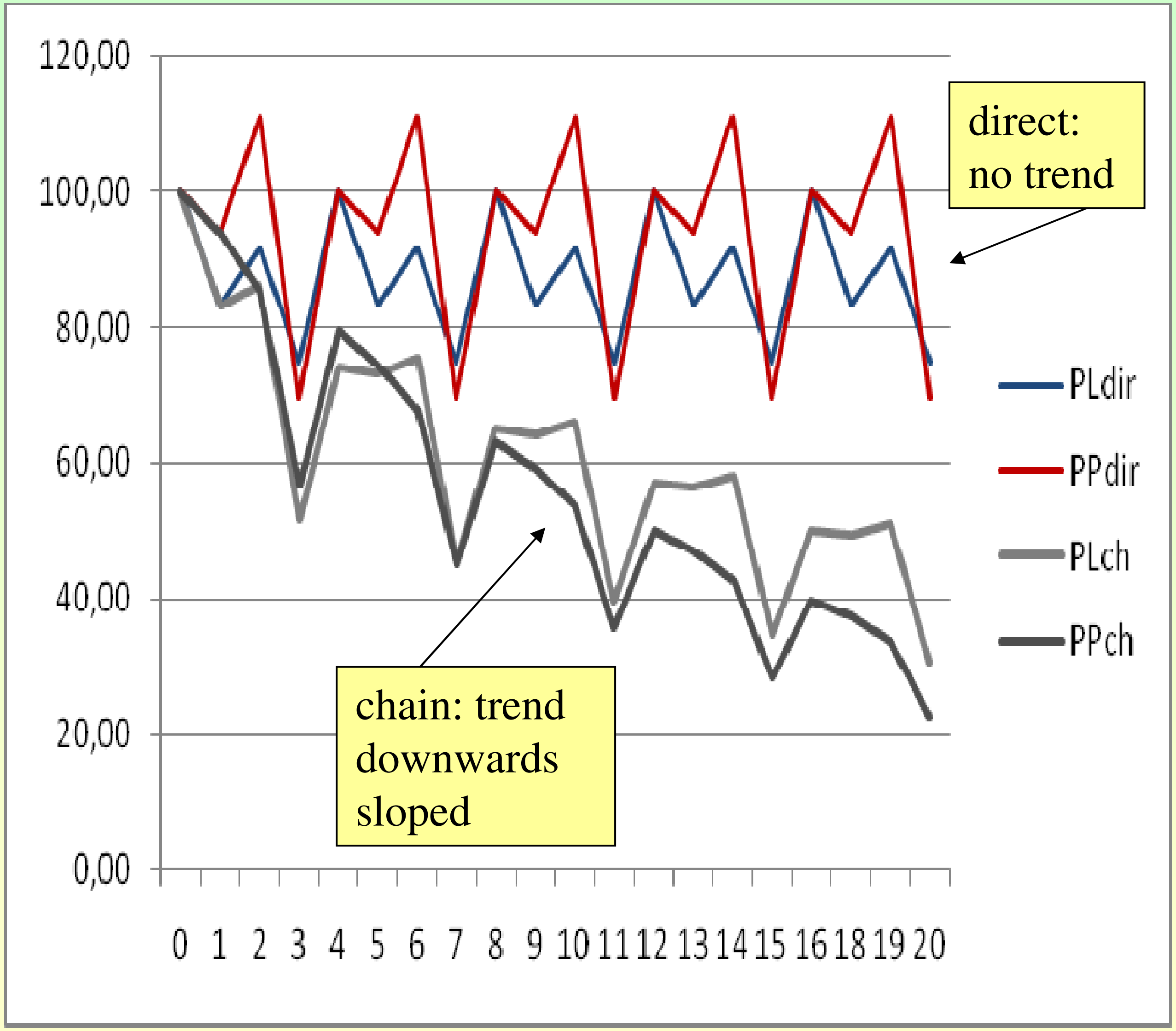


Here direct price indices of Laspeyres and Paasche (in principle declining prices) → next slide: **chain price indices** → and then: consequences for the direct **quantity** indices (given the value index)

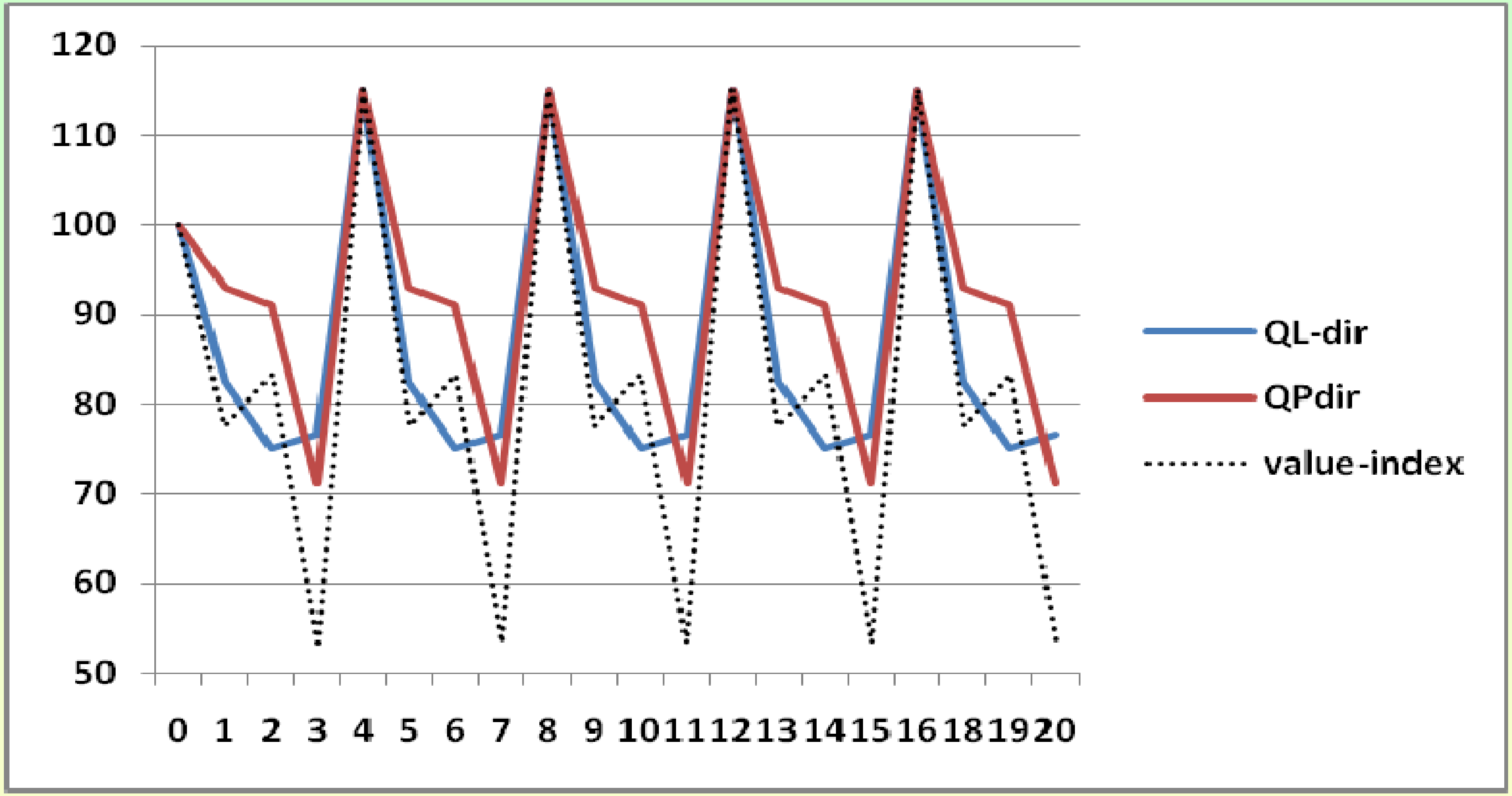
3.5.2 (2) Price indices (direct and chain) in the case of cyclical movement

PLdir	PPdir	PLch	PPch
100,00	100,00	100,00	100,00
83,33	93,94	83,33	93,94
91,67	111,11	86,02	85,40
75,00	69,57	51,61	56,93
100,00	100,00	74,19	79,36
83,33	93,94	73,12	74,55
91,67	111,11	75,48	67,77
75,00	69,57	45,29	45,18
100,00	100,00	65,10	62,98
83,33	93,94	64,16	59,16
91,67	111,11	66,22	53,79
75,00	69,57	39,73	35,86
100,00	100,00	57,12	49,98
83,33	93,94	56,29	46,95
91,67	111,11	58,11	42,69
75,00	69,57	34,86	28,46
100,00	100,00	50,12	39,67
83,33	93,94	49,39	37,26
91,67	111,11	50,98	33,88
75,00	69,57	30,59	22,58

"bouncing" of price indices when oscillation takes place



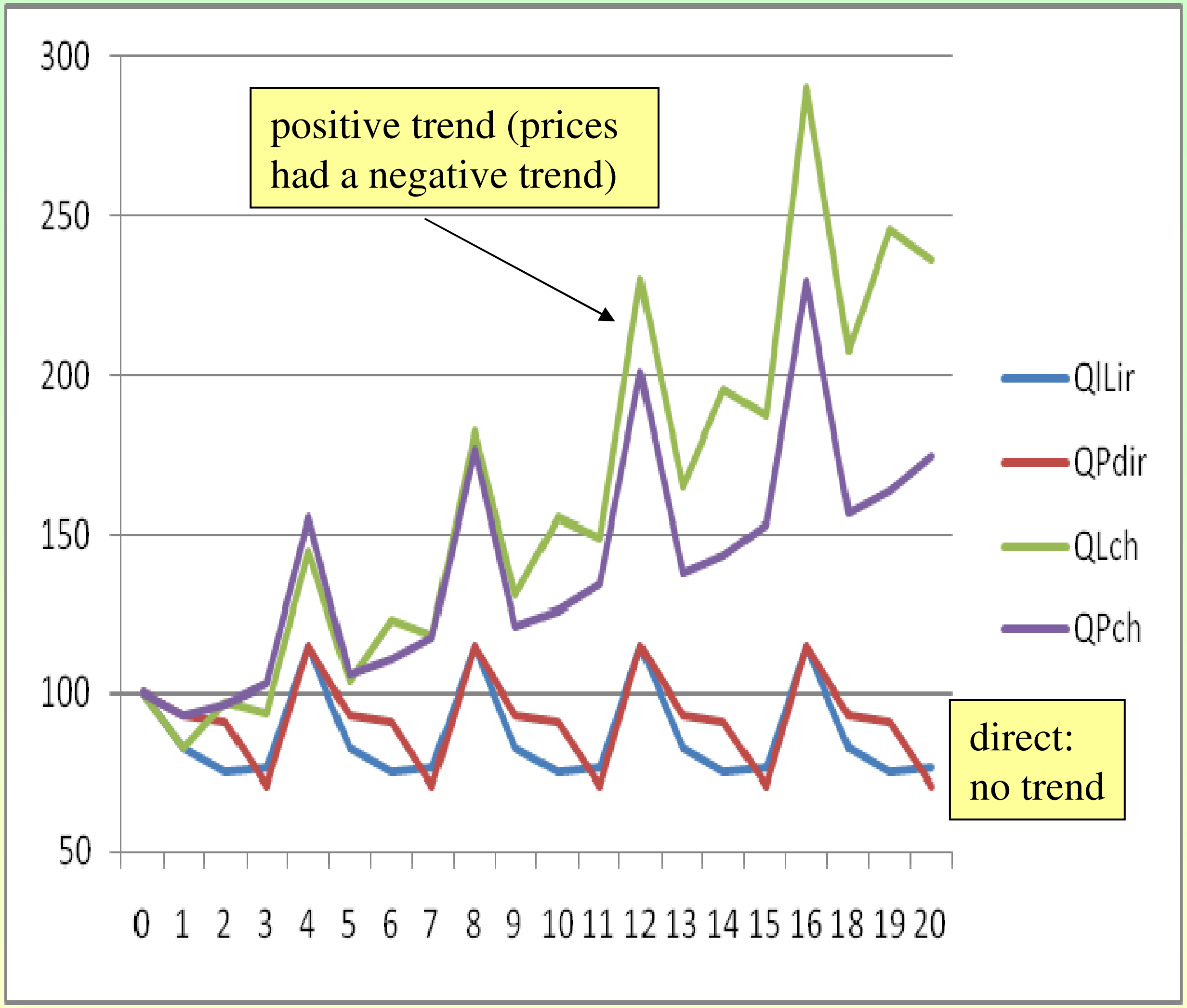
3.5.2 (3) value index and quantity indices



$$V_{0t} = P_{0t}^{L} Q_{0t}^{P} = P_{0t}^{P} Q_{0t}^{L} \qquad V_{0t} = \overline{P}_{0t}^{L} \overline{Q}_{0t}^{P} = \overline{P}_{0t}^{P} \overline{Q}_{0t}^{L}$$

3.5.2 (4) The same situation: implicit **quantity** indices (direct and chain)

QPdir	QIdir	QPch	QLch
100,00	100,00	100,00	100,00
93,00	82,50	93,00	82,50
90,91	75,00	96,88	97,58
71,11	76,67	103,33	93,68
115,00	115,00	155,00	144,91
93,00	82,50	105,99	103,96
90,91	75,00	110,41	122,96
71,11	76,67	117,77	118,04
115,00	115,00	176,65	182,59
93,00	82,50	120,80	130,99
90,91	75,00	125,83	154,93
71,11	76,67	134,22	148,74
115,00	115,00	201,33	230,08
93,00	82,50	137,68	165,06
90,91	75,00	143,41	195,23
71,11	76,67	152,97	187,42
115,00	115,00	229,46	289,91
93,00	82,50	156,91	207,98
90,91	75,00	163,45	246,00
71,11	76,67	174,35	236,16



chain quantity indices are constantly rising just as chain price indices are declining

3.5.2 (5) The example in terms of growth rates and correlations

$g_t^0 = P_t^L$	$g_t^{t-1} = \overline{P}_t^L$	Difference*	$P_{03}^L = g_1^0 \ldots g_t^0$	$\overline{P}_{03}^L = g_1^0 g_2^1 \ldots g_t^{t-1}$
$g_1^0 = 0.8333$	$g_1^0 = 0.8333$	0	0.8333	0.8333
$g_2^0 = 1.1$	$g_2^1 = 1.0323$	-0.0677	$P_{02}^L = g_1^0 g_2^0 = 0.9167$	$\overline{P}_{02}^{LC} = g_1^0 g_2^1 = 0.8602$
$g_3^0 = 0.8182$	$g_3^2 = 0.6$	-0.2182	$P_{03}^L = 0.75$	$\overline{P}_{03}^L = 0.5161$
$g_4^0 = 4/3 = 1.33$	$g_4^3 = 1.4375$	+0.1042	$P_{04}^L = 1.0$	$\overline{P}_{04}^L = 0.7419$
$g_5^0 = g_1^0$	$g_5^4 = 0.9856$	+0.1523		
$g_6^0 = g_2^0$	$g_6^5 = g_2^1$	-0.0677		

* related to the temporal covariance

$$D_{0t}^{PL} = \frac{\overline{P}_{0t}^L}{P_{0t}^L} = \frac{g_1^0 g_2^1 g_3^2 \ldots g_t^{t-1}}{g_1^0 g_2^0 g_3^0 \ldots g_t^0}$$

$$\text{if } \frac{g_t^{t-1}}{g_t^0} > 1, \text{or } g_t^{t-1} - g_t^0 > 0 \Rightarrow D_{0t}^{PL} > D_{0,t-1}^{PL}$$

$D_{01}^{PL} = 1$ $\quad D_{02}^{PL} = 0.6882$

$D_{03}^{PL} = 0.742$ $\quad D_{04}^{PL} = 0.877$

3.5.3 (1) More about drift, LPG (PLS) and temporal covariance

1. From $\overline{P}^{C}_{0t} = \overline{P}^{C}_{0s}\overline{P}^{C}_{st}$ follows $D^{PL}_{0t}P^{L}_{0t} = \left(D^{PL}_{0s}P^{L}_{0s}\right)\left(D^{PL}_{st}P^{L}_{st}\right)$

The drift function is <u>not</u> transitive $D^{PL}_{0t} \neq D^{PL}_{0s}D^{PL}_{st}$

$$\frac{D^{PL}_{0t}}{D^{PL}_{0s}D^{PL}_{st}} = \frac{P^{L}_{0s}P^{L}_{st}}{P^{L}_{0t}}$$

2. To D the antithetic Paasche-Laspeyres relation applies $D^{PL}_{0t} = \dfrac{1}{D^{QP}_{0t}}$

3. Paasche-drift more complicated than Laspeyres -drift

$$D^{PL}_{0t} = \frac{g_2^1 g_3^2 \cdots g_t^{t-1}}{g_2^0 g_3^0 \cdots g_t^0} = \prod_{\tau=2}^{\tau=t} \frac{g_\tau^{\tau-1}}{g_\tau^0}$$

$$D^{PP}_{0t} = \frac{g_1^1 g_2^2 \cdots g_{t-1}^{t-1}}{g_2^t g_3^t \cdots g_{t-1}^t} = \prod_{\tau=1}^{\tau=t} \frac{g_\tau^\tau}{g_\tau^t}$$

4. All relevant elements in the G-matrix $\mathbf{G} = \begin{bmatrix} \mathbf{g}' \\ \mathbf{G}^* \end{bmatrix}$

$$\mathbf{g}' = \begin{bmatrix} g_1^0 & g_2^0 & g_3^0 & g_4^0 \end{bmatrix}$$

$$\mathbf{G}^* = \begin{bmatrix} g_1^1 & g_2^1 & g_3^1 & g_4^1 \\ g_1^2 & g_2^2 & g_3^2 & g_4^2 \\ g_1^3 & g_2^3 & g_3^3 & g_4^3 \\ g_1^4 & g_2^4 & g_3^4 & g_4^4 \end{bmatrix}$$

$$P^{L}_{0t} = g_1^0 g_2^0 g_3^0 \ldots$$

$$\overline{P}^{L}_{0t} = g_1^0 g_2^1 g_3^2 \ldots$$

$$P^{P}_{03} = g_1^3 g_2^3 g_3^3$$

$$\overline{P}^{P}_{0t} = g_1^1 g_2^2 g_3^3 \ldots$$

5. We now redefine the Laspeyres-Paasche gap and make use of a theorem of L. v. Bortkiewicz

3.5.3 (2) More about LPG (PLS) and Bortkiewicz's theorem (of the covariance)

The Laspeyres Paasche gap LPG as ratio

direct index $\eta_{0t} = \frac{P^L_{0t}}{P^P_{0t}} = \frac{\gamma_{0t}}{P^P_{0t}} + 1$ chain index $\overline{\eta}_{0t} = \frac{\overline{P}^L_{0t}}{\overline{P}^P_{0t}} = \frac{\overline{\gamma}_{0t}}{\overline{P}^P_{0t}} + 1$

rather than as a difference* $\gamma_{0t} = P^L_{0t} - P^P_{0t}$ and $\overline{\gamma}_{0t} = \overline{P}^L_{0t} - \overline{P}^P_{0t}$

t	$\overline{\eta}_{0t}$	The relevant linear indices
1	g^0_1 / g^1_1	$g^0_1 = P^L_{01}, \quad g^1_1 = P^P_{01}$
2	$(g^0_1 / g^1_1) \cdot \frac{g^1_2}{g^2_2}$	$g^1_2 = \frac{\sum p_2 q_1}{\sum p_1 q_1}, \; g^2_2 = \frac{\sum p_2 q_2}{\sum p_1 q_2}$

The well known special case of the theorem: all depends on the covariance between price and quantity relatives: if cov < 0 then $P^P < P^L$

covariance between p_2/p_1 and q_2/q_1 (weights $p_1q_1/\Sigma p_1q_1$):
if cov < 0 (that is $P^L > P^P$ gap will **widen**

* the two gaps are related as follows $\overline{\gamma}_{0t} = \gamma_{0t} - \left[P^L_{0t}\left(1 - DP^{PL}_{0t}\right) - P^P_{0t}\left(1 - DP^{PP}_{0t}\right)\right]$

3.5.3 (3) More about drift, LPG (PLS) and temporal covariance

Rules for the LPGs $\overline{\eta}_{0t} = \frac{\overline{P}^L_{0t}}{\overline{P}^P_{0t}}$ and $\eta_{0t} = \frac{P^L_{0t}}{P^P_{0t}}$

Because of the Paasche formula a theory of the gap is more difficult than about the drift

and the drift[a)] $D^{PL}_{03} = \frac{g_2^1 g_3^2 \ldots g_t^{t-1}}{g_2^0 g_3^0 \ldots g_t^0}$

All statements are derived from the theorem of Ladislaus von Bortkiewicz (sec. 5)

term	equation for the **change** of the term[b)]	The relevant covariance and interpretation
gap chain	$\overline{\eta}_{0,t-1} \cdot \frac{g_t^{t-1}}{g_t^t}$	If covariance between price relatives p_t/p_{t-1} and quantity relatives q_t/q_{t-1} (weights $p_{t-1}q_{t-1}/\Sigma\, p_{t-1}q_{t-1}$) is negative: gap widens
drift Lasp. prices	$D^{PL}_{0,t-1} \frac{g_t^{t-1}}{g_t^0}$	If covariance between price relatives p_t/p_{t-1} and quantity relatives q_{t-1}/q_0 (weights $p_{t-1}q_0/\Sigma\, p_{t-1}q_0$) is negative: drift will increase
gap direct	no simple relation between	$\eta_{0,t-1} = \frac{g_1^0 g_2^0 \ldots g_{t-1}^0}{g_1^{t-1} g_2^{t-1} \ldots g_{t-1}^{t-1}}$ and $\eta_{0t} = \frac{g_1^0 g_2^0 \ldots g_{t-1}^0 g_t^0}{g_1^t g_2^t \ldots g_{t-1}^t g_t^t}$

a) Formula for Paasche is difficult

b) there are only formulas for the change of …

3.6 (1) The notion of pure price/quantity comparison

Non-chainers criticize chain indices mainly because they do not provide a "pure" comparison; in the following dimensions:

Chain price indices violate "pure" [a] price comparison in the sense of not **only** being affected by

prices	**periods 0 and t**
but also by changes in (the structure of) quantities (weights), qualities, types of products, outlets etc.[b]	but also by referring to other periods and depending on the path connecting 0 and t (not only on the endpoints 0 and t) [c]
no elimination of structural change [d]	path dependence (no chainability)

a) "Pure" means that situations to be compared should differ in only *one* aspect in order to avoid difficulties (ambiguities) of interpretation and to *make sure that like is compared with like*.

b) this applies to unit value indices as well for example

c) as the first aspect (i.e. prices) refers to the aggregation over commodities, this (second) notion of "pure" refers to the temporal aggregation (over intervals in time)

d) see next page for why it is essential to eliminate the structural change

3.6 (2) The notion of pure price/quantity comparison

Why price **index** (or wage index) and not only **average** prices (or wages respectively)?

Imagine an economy with only two industries A and B, and wages of €10 and €16 paid at base period:

situation in base period			
industry	wage	hours	payment
A	10	50	500
B	16	50	800
sum*	13	100	1300

* or average

In t all wages have been raised in unison by 50%

wage	hours	payment
15	90	1350
24	10	240
15.9	100	1590

remember SNA about unit value indices

It would not make sense to compare simply the **average wage** per hour (13€ and 15.90 €) and conclude that wages rose only by 22.3% (15.93/13) because the structure changed in favour of the low-wage-level industry A (22.3% < 50%).

Values and averages are affected by structures

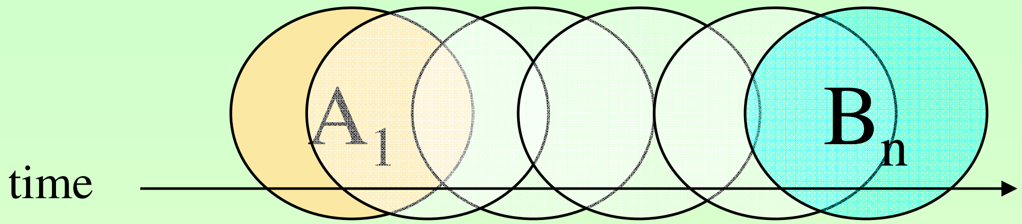
$$V_{0t} = \frac{\sum p_t q_t}{\sum p_0 q_0} = \underbrace{\frac{\sum p_t q_0}{\sum p_0 q_0}}_{\text{pure}} + \underbrace{\frac{\sum p_t (q_t - q_0)}{\sum p_0 q_0}}_{\text{structure}}$$

There were no need for indices if

- the structure would remain constant and
- we always would compare only two adjacent periods (no time series)

3.6 (3) "Comparability" in the context of direct- and chain-comparison

The solution to the "multiplication mystery"



What is directly incomparable is nonetheless indirectly comparable

Solution to the mystery: The underlying definition of "comparability is different"

This can be seen by asking	direct	indirect
1. how much A and B must have in common in order to be comparable	strictly speaking every-thing except time of recording	possibly nothing (if only there are some overlapping links)
2. For how long an interval a comparison can reasonably made?	only over very short intervals unless the structure remains const.	no limitation for the length of the interval
3. the result	is never path dependent	is path dependent

4. For which t (t = 1, …,n) A_t is no longer A_t but B_t? (where is the criterion for differentiation?)

3.6 (4) The meaning of "pure" comparison in the case of an index (price index)

more about this v.d.Lippe (2001) ch. 8.2: There I made a distinction between three concepts; here I introduce only one concept (P1) and additional desirable properties (P2)

	includes	rules out
P1 Successive price indices should **differ only with respect to prices** (ceteris paribus) [a)]	all unweighted direct indices; as weighted indices: Laspeyres; geometric, harmonic, or quadratic mean etc. using base period expenditure weights s_{i0}	Paasche, Fisher, Walsh,… (all superlative indices because they make use of q_0 [constant] *and* q_t [variable]); all chain indices
P2(a) index should have a **ratio-of-expenditures** interpretations and **P2(b)** should be **linear in the prices p_t** [b)]	P2(a) rules out all unweighted indices and indices with weights not related to quantities both P2(a) and P2(b) are not fulfilled in the case of a harmonic, quadratic, or geometric (= log-Laspeyres index) mean of price relatives weighted with s_{i0} only P^L is left	

a) all elements of a price index formula other than prices are kept constant (these are the weights [which are not necessarily related to quantities])

b) differences in the index values can be accounted for differences in the prices of certain goods

3.6 (5) Common misunderstanding of why a "fixed basket" is assumed

As to the popular derision of the idea of a fixed basket: the reason is possibly that an **analytical device is mistaken for a statement describing the real world**.

To assume a **fixed basket for analytical purposes** does not mean that consumption is not responding to changing prices, or that the economy is static. The Laspeyres approach should not be ridiculed with arguments like "The American economy is flexible and dynamic."* There is no need to deny this if you favour P^L.

The fixed basket is **a model like the model of a "life table"** (or stationary) **population** in which **death risk** depending on age is kept **constant for ≈ 100 years** (same age = same risk, irrespective of the birth-cohort to which one belongs).

Without *such a model* though *clearly in contradiction to observation* and real world conditions (such as increased longevity as a result of progress in medicine) measurement of life expectancy would be impossible.** It is nonsense to say, life expectancy were incorrectly or "inaccurately" measured because the assumptions of the underlying model are unrealistic.

* Final Report of the "Boskin Commission"

** Life expectancy cannot be measured by asking people how long they expect to live

3.7 (1) Summary: Review of the critique of arguments in favour of chain indices

1. Justification of chain indices not theory-driven
   inconsistency (unit value indices), theorem of Funke, one-sidedness (no disadvantages mentioned), substitution bias (why not direct P^F?)
2. Advantages mainly derived from a critique of the *fixed basket* (direct *Laspeyres*) approach;
   they do not apply to certain "superlative indices" like P^F
3. "Solution vs. dissolution"
   e.g. choice of base period, quality adjustment
4. should be advantageous especially in those cases in which comparisons with direct indices fail
   that is over particularly *long* intervals in time whenever consumption patterns change *rapidly* and *fundamentally* (but are they really fit for just *such* situations?)
5. Most recent weights not necessarily the most "relevant" and most "representative"
   Two assumptions tacitly made

3.7 (2) Summary: Review of the critique of arguments in favour of chain indices

6. Most arguments in favour of chain indices are not tenable
   Many implicitly take the link for the chain, or mystify the simple fact of multiplying links
7. Chain indices have poor axiomatic properties: they fail identity and other axioms;
   alleged advantages of a certain link formula as compared to another have little relevance: axioms apply to links only not to the chain
8. They have in particular poor aggregation properties
   regarding both, time aggregation and aggregation over commodities (sub-indices); chain indices may in particular suffer from path dependence
9. When applied to deflation there are (new) problems with additivity and integrating QNA in ANA
   both problems are consequences of applying chaining on indices that are not transitive (= consistently aggregative over time)
   practice of NSI publications no longer uniform ("real" aggregates)
10. More demanding as regards data (updating of weights)
    more difficult to compare different indices (as e.g. productivity measurement, terms of trade,"real" income etc) when all indices are chain indices ⇒

3.7 (3) What happens when all indices are chain indices?

Once chain indices are introduced for CPIs and deflation there is a strong temptation to make use of this principle in *all* kinds of indices, also production indices, indices of new orders and the like. Given problems with aggregation and path dependence:

Our question: Have we sufficiently considered the *impact on the analysis of*

(1) Statistics defined as **relations between** (e.g. ratios of) **two indices**, e.g. "terms of trade", "productivity", "real wages" etc. ?

(2) Methods combining two or more indices and implicitly assuming additivity like for example **double deflation** (→ sec. **6.1**) ?

(3) using indices in order to define growth rates, endpoints of intervals, **turning points**, **leads/lags**, "dating" phases of the business cycles etc. ?

Will "turning points" diagnosed with chain rather than direct indices be more reliable?

Chain indices are, however, acceptable or even commendable if

- "pure" price/quantity comparison is not found essential, and
- other aspects are found more important, as e.g.

1. to approximate a superlative index (reduce the LPG)
2. to have less difficulties with emergence of new goods or disappearance of old ones (or: to accommodate with a changing domain of definition)

It is not guaranteed that LPG will be reduced, and in 1 superiority of "superlative" indices is tacitly assumed

for Fisher see section 5

2 should give rise to another interpretation of the index*

1 and 2 may be justified using COLI Theory

* no longer cost for a given basket or utility level

References (1)

Eiglsperger, Martin and **Schackis**, Daniela, Weights in the Harmonised Index of Consumer Prices: Selected Aspects from a User's Perspective, Contribution to the Meeting of the Ottawa Group in Neuchatel, Switzerland 2009

Hill, Robert J. (2006), When Does Chaining Reduce the Paasche-Laspeyres Spread? An Application to Scanner Data, Review of Income and Wealth, Series 52, No. 2, pp. 309 - 325

Ivancic, Lorraine, **Fox**, Kevin, J. and **Diewert**, W. Erwin, Scanner Data, Time Aggregation and the Construction of Price Indexes, Contribution to the ….

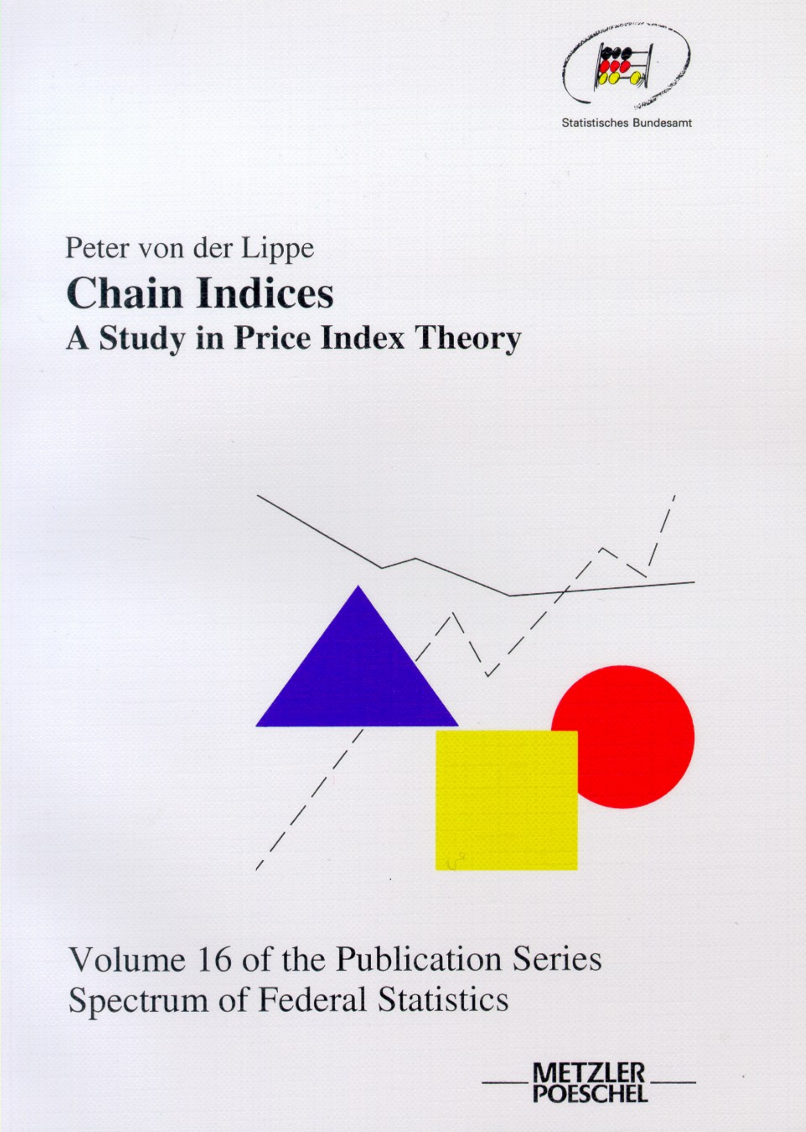
Schreyer, Paul, Chain Index Number Formulae in the National Accounts, ppt-presentation in an OECD-NBS Workshop, 6-10 Dec. 2004

Tödter (2005), Karl-Heinz, Umstellung der Deutschen VGR …, Deutsche Bundesbank, Working Paper (series 1) 31/2005 (Diskussionspapier, Reihe 1, Volkswirtschaftl. Studien)

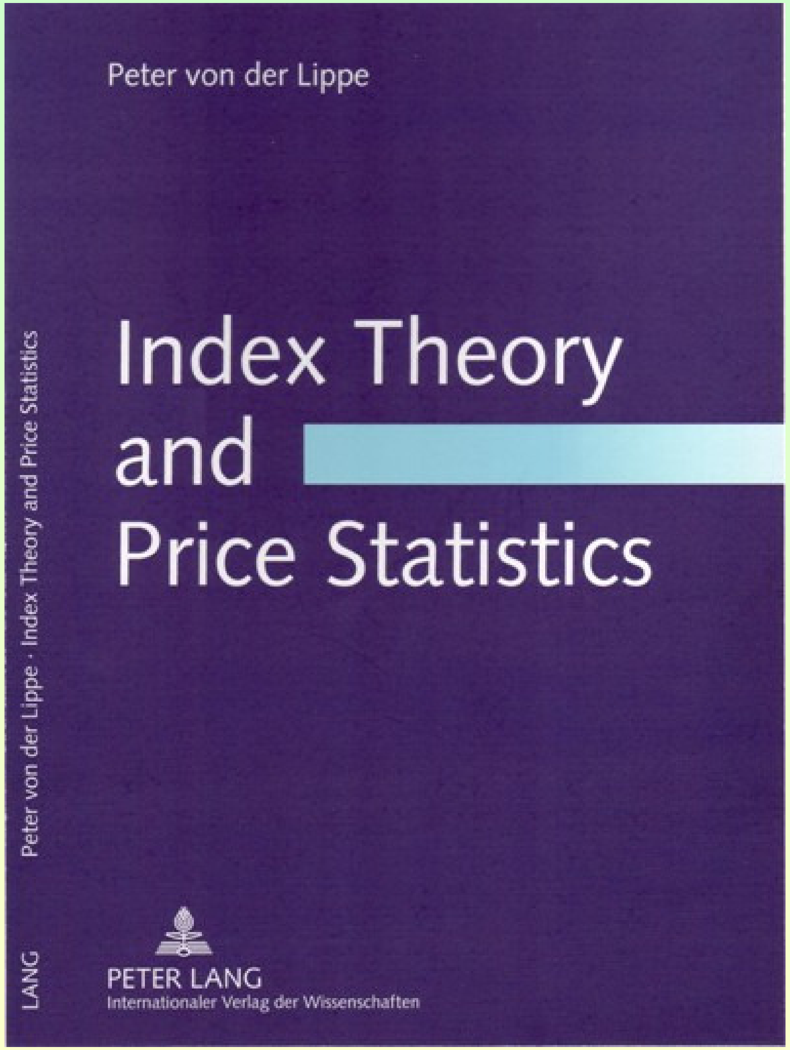
von der Lippe (2001), Peter, Chain Indices, A Study in Index Theory, Fed. Statistical Office Germany, Wiesbaden (publisher Metzler-Poeschel, Stuttgart)

von der Lippe (2007), Peter, Index Theory, Peter Lang, Frankfurt etc.

References (2) Finally: some heretical literature



Stuttgart 2001, ISBN3-8246-0638-0



Frankfurt/M. etc. 2007,
ISBN 978-3-631-56317-5