UNIVERSITÄT DUISBURG ESSEN

Prof. Dr. Peter von der Lippe
www.von-der-lippe.org

Problems with Chain Indices (II)

Implementation, Aggregation and Deflation

Course delivered at the European Central Bank Frankfurt

4. Chain indices everywhere: the triumph of chainers

- 4.1 Regulations and projects: European Union*
- 4.2 Other countries (USA)*
- 4.3 Experiences, empirical findings

5. Aggregation and "additivity"

- 5.1 Types of aggregation and "additivity"
- 5.2 Additivity and linearity; theorems on linear indices
- 5.3 Fisher's "ideal" index is far from ideal

6. Deflation and chain indices

- 6.1 Task of deflation
- 6.2 Notion of "volume"
- 6.3 Criteria for "good" deflation
- 6.4 Direct and chain price indices as deflators
- 6.5 How to deal with non-additivity of volumes?

* This part of the course is necessarily only incomplete and provisional

4.1.1 (1) European Union: Regulations on HICP weights

Quotations relating to these regulations on the next slide

Selected HICP Regulation dealing inter alia with chain index issues
A = Council Regulation , B = Commission Regulation

Nr., date, type	Contents
A **2494/95** 23 Oct. 1995	defines aim, **comparability**; timetable, procedure etc. of harmonization but no details of compilation of indices (more
B **1749/96** 9 Sept. 1996	Initial coverage of goods and services, practices for updating the coverage and inclusion of **newly significant goods and services**
B **2454/97** 10 Dec. 1997	concerning **minimum standards for the quality of HICP weights** • defines a maximum age of weights (7 years) and, • requires an annually checking of "critical" weights
B **1921/2001** 28. Sept. 2001	Standards for **revisions of the HICP** (revisions have to be approved and there is no quantitative assessment of the impact of revisions unless a revision affects the results by more than **1 per thousand**)
B **1708/2005** 19. Oct. 2005	Index reference period, amending No 3 (2214/96 temporal coverage of price collections), introducing **consumption** "**segments**"* with far reaching implications for quality adjustment and replacement strategy

*serving the same purpose from the point of view of households

4.1.1 (2) Regulations on HICP weights: More details of some regulations

A **2494/95** 23 Oct. 1995	"HICPs shall be considered to be **comparable if** they reflect only differences in price changes or consumption patterns between countries. HICPs which differ on account of differences in the concepts, methods or practices used in their definition and compilation shall not be considered comparable." "…. more than ***0.1 percentage point*** on average over one year against the previous year cannot be accepted."
B **1749/96** 9 Sept. 1996	"**Newly significant goods and services (NSG)** are defined as those goods and services the price changes of which are not explicitly included in a Member State's HICP and the estimated consumers' *expenditure on which* has become at least ***one part per thousand*** of the expenditure covered by that HICP." Compulsory checks (once a Member State reports NSG) and adjustments
B **2454/97** 10 Dec.1997	***maximum age of weights*** …weightings which reflect consumers' expenditure patterns in a weighting reference period ending ***no more than seven years*** before ***frequency of revision*** Each year, Member States shall carry out a review of weightings in order to ensure that they are sufficiently reliable and relevant ***obligatory adjustment of weights*** Where reliable evidence shows ... [that a weighting change] ... would affect the change in the HICP by more than ***0.1 percentage point*** on average over one year against the previous year Member States shall adjust the weightings of the HICP appropriately

4.1.1 (3) HICP Formula: 1. the national indices H

The formula below represents the planned state when the dates of weights will be harmonised (see slide 7)

December [month 12] of year t-1 is the linking month of this chain-linked Laspeyres type index. For this purpose weights (of National Accounts) are "price updated" only (as a rule volumes are less frequently updated) and normalized (in order to sum up to unity)

$$H_{0t,m} = \frac{\sum p_{t,m} q_{t-2}}{\sum p_{t-1,12} q_{t-2}} \frac{\sum p_{t-1,12} q_{t-3}}{\sum p_{t-2,12} q_{t-3}} \left(\frac{\sum p_{t-2,12} q_{t-4}}{\sum p_{t-3,12} q_{t-4}} \cdots \right)$$

t,1 → t,m

t = 0 → t-1, m=12

$$H_{0,t-1,m} = \frac{\sum p_{t-1,m} q_{t-3}}{\sum p_{t-2,12} q_{t-3}} \left(\frac{\sum p_{t-2,12} q_{t-4}}{\sum p_{t-3,12} q_{t-4}} \cdots \right)$$

t-1,1→ t-1,m

t = 0→ t,-2,m=12

Obviously the expressions in brackets will cancel out when a ratio of two price indices, both for a month m is formed

and his ratio then is given by

$$H_{0t,m} / H_{0,t-1,m} = \frac{\sum p_{t,m} q_{t-2}}{\sum p_{t-1,12} q_{t-2}} \bigg/ \frac{\sum p_{t-1,m} q_{t-3}}{\sum p_{t-2,12} q_{t-3}}$$

current year

past years

note: **two** baskets (q-vectors) involved

4.1.1 (4) HICP Formula: 2. the multi-national index M as average of H-indices

Comparing month m in t with m in the previous year

$$H_{0t,m}/H_{0,t-1,m} = \frac{\sum p_{t,m}q_{t-2}}{\sum p_{t-1,12}q_{t-2}} \Big/ \frac{\sum p_{t-1,m}q_{t-3}}{\sum p_{t-2,12}q_{t-3}} = \frac{\sum \frac{p_{t,m}}{p_{t-1,12}} \cdot w_{t-1}}{\sum \frac{p_{t-1,m}}{p_{t-2,12}} \cdot w_{t-2}}$$

← same formula as on preceding slide

implies two weighting structures $w_{t-1} = \frac{p_{t-1,12}q_{t-3}}{\sum p_{t-1,12}q_{t-3}}$ and $w_{t-1} = \frac{p_{t-1,12}q_{t-3}}{\sum p_{t-1,12}q_{t-3}}$

The national indices H are combined to the multinational index M using country weights c_m

$$M_{05} = \left(\sum c_{m0}H_{m01}\right)\left(\sum c_{m1}H_{m12}\right)\left(\sum c_{m2}H_{m23}\right)\left(\sum c_{m3}H_{m34}\right)\left(\sum c_{m4}H_{m45}\right)$$

the summation takes place over m = 1, …, M member countries

M is affected by

- the *prices* in each member country in each period,
- changing weights of the *commodities*,
- changing domain of definition (new products, outlets etc.) in each country and each period, and
- the *path* of the index since a chain index is always depending on its "history"
- (varying) country weights c_m (and number M).

4.1.2 (1) European Union: Projects, discussions concerning the HICP (overview)

This section deals with ongoing discussions about HICP methods and problems stemming from the chain-index approach of the HICP. It is necessarily incomplete and should be up-dated with the passage of time. Such topics are

1. Harmonization of the practice of establishing HICP weights

- The present practice allows weights of an age up to seven years is widely different across Member States (MS)
- In some MS weights are derived from HES (as the only reliable source for detailed weights) in other MS from NA
- In which detail and which frequency weights (inclusive of quantities) are to updated?
- How a uniform and more frequent update should be carried out in practice?
- no longer weights of different age

2. Relevance, meaning and method of (isolated) price updating

Is it correct to say – as often maintained - that price updating only (without updating quantities) is inherent in the Laspeyres (fixed base) approach?

If there is only one month as linking period for updating of prices there may be problems with goods like package holiday: is one month representative? are seasonally adjusted or unadjusted prices to be used?

4.1.2 (2) Tighter regulations on HICP weights: Present situation, projects

Eiglsperger/Schackis:

"The actual **practices** of updating weights **differ** across the national institutes compiling HICPs, ranging from annual updates to general reviews of weights conducted in five year intervals. These different practices have been made congruent for HICP purposes in order to allow national HICPs to be aggregated, but only in formal terms, i.e. by introducing a price-updating of weights to the December of the respective previous year."

majority of MS review annually HICP sub-index weights on the basis National Accounts

less detailed then HES and subject to revisions

Austria, Belgium, Cyprus, Denmark, Finland, Germany, Greece, Ireland and Malta conduct a general update of volumes at three to five years intervals using HES* data (not annually available)

* household expenditure surveys

Projects, new initiatives (Eurostat 2008)

Speedier and more uniform (tighter standards for the) revision of weights. More frequent updates are found necessary esp. in the case of fast evolving markets (e.g. information and communication technology)

Amendment or regulation 2454/97

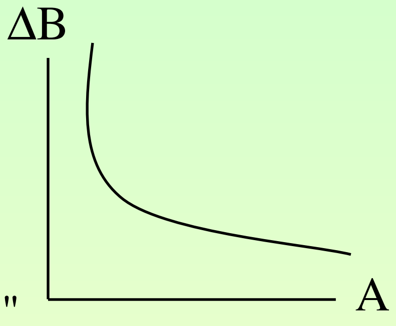
4.1.2 (3) Strategies in updating weights

Annual update not only of prices but also quantities in the weights is considered desirable. Such weights may, however, be not reliable or too costly. Therefore a *strategy of updating*:

Ideas: 1. not all items (weights) equally relevant, 2. case by case approach, and 3. quantities in weights can be more or less prone to shifts

1) impact of 1 per 1000 only if

$$\frac{p_{it}}{p_{i0}}\left(\frac{p_{i2}q_{i2}}{\sum p_{i2}q_{i2}}-\frac{p_{i1}q_{i1}}{\sum p_{i1}q_{i1}}\right)=A\cdot\Delta B>0.001$$



A **and** ΔB must be >> 0. Eurostat: "fairly insensitive to changes in weights"

2) case-by-case: weights should reflect current consumption patterns; t-2 (for quantities) as a compromise (in view of the resources needed for updating), however, weights need not have the same age because:

3) **types of weights**

critical because of structural **shifts**: Health care (reforms), goods with administered prices

possibly irregular movement of prices and/or quantities → smoothing of weights? quick switch to new weights perhaps not desirable

"non-critical" = less prone to structural shifts (e.g. non-durable goods) smoothly evolving new weights

4.1.2 (4) Regulations and projects: The price updating

In discussion about a HICP price-updating (December) it is popular to refer to the product representation of the direct Laspeyres price index

$$\overline{P}^{L}_{03} = \left(\sum \frac{p_1}{p_0}\frac{p_0q_0}{\sum p_0q_0}\right)\left(\sum \frac{p_2}{p_1}\frac{p_1q_1}{\sum p_1q_1}\right)\left(\sum \frac{p_3}{p_2}\frac{p_2q_2}{\sum p_2q_2}\right)$$

It is maintained that the direct P^L is also a product

$$P^{L}_{03} = \left(\sum \frac{p_1}{p_0}\frac{p_0q_0}{\sum p_0q_0}\right)\left(\sum \frac{p_2}{p_1}\frac{p_1q_0}{\sum p_1q_0}\right)\left(\sum \frac{p_3}{p_2}\frac{p_2q_0}{\sum p_2q_0}\right) = \frac{\sum p_3q_0}{\sum p_0q_0}$$

so a regular price **update is cogent even in** a **direct** P^L **approach**

However

these terms are **not in use in the direct approach** and **they have to be chain-linked**

this formula is not gained by multiplying: **there is no chain-linking** and

they are re-based indices $P^{L}_{02(1)} = P^{L}_{02} / P^{L}_{01}$ and $P^{L}_{03(2)} = P^{L}_{03} / P^{L}_{02}$

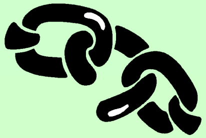
care has to be taken for matching

The direct index can be written in both ways (ratio *and* product), the chain index can *only* be written (and **compiled**) as a product

no care …

4.1.2 (5) Price updating of weights: why and how

Moreover: chain indices are affected by



direct indices not (also basket interpretation)

Therefore:

"Price-updating is __inherent__ in the definition of the Laspeyres price index"

is <u>not</u> correct.

It disregards all differences between chain indices and direct indices

How to price-update HICP expenditure weights?

A = any constant period

on the level of price relatives (elementary indices)

$$\frac{p_{i0}q_{iA}}{\sum p_{i0}q_{iA}} \rightarrow \frac{\sum p_{i1}q_{iA}}{\sum\sum p_{i1}q_{iA}}$$

$$\boxed{p_{i1}q_{iA} = p_{i0}q_{iA} \cdot \left(\frac{p_{i1}}{p_{i0}}\right)}$$

and subsequent summation

on the level of sub indices of the price index

$$\frac{\sum p_{i0}q_{iA}}{\sum\sum p_{i0}q_{iA}} \rightarrow \frac{\sum p_{i1}q_{iA}}{\sum\sum p_{i1}q_{iA}}$$

$$\boxed{\sum p_{i1}q_{iA} = \sum p_{i0}q_{iA}\left(\frac{\sum p_{i1}q_{iA}}{\sum p_{i0}q_{iA}}\right)}$$

and summation to ΣΣ

The sub-index (term in brackets) is used to update (the respective weight)

4.1.2 (6) Different lags of prices and quantities (1)

To arrive at a genuine Laspeyres chain (price) index

$$\overline{P}^{L}_{0t}=\left(\sum\frac{p_1}{p_0}\frac{p_0q_0}{\sum p_0q_0}\right)\left(\sum\frac{p_2}{p_1}\frac{p_1q_1}{\sum p_1q_1}\right)\left(\sum\frac{p_3}{p_2}\frac{p_2q_2}{\sum p_2q_2}\right)\ldots.=\frac{\sum p_1q_0}{\sum p_0q_0}\frac{\sum p_2q_1}{\sum p_1q_1}\frac{\sum p_3q_2}{\sum p_2q_2}\ldots$$

it is crucial to have price and quantity updates at the same intervals.

Some countries suggest that t-2 expenditures should be taken directly as an estimate for t-1 expenditures. *Without* price updating this amounts to

$$\breve{P}^{L}_{0t}=\left(\sum\frac{p_1}{p_0}\frac{p_{-1}q_{-1}}{\sum p_{-1}q_{-1}}\right)\left(\sum\frac{p_2}{p_1}\frac{p_0q_0}{\sum p_0q_0}\right)\left(\sum\frac{p_3}{p_2}\frac{p_1q_1}{\sum p_1q_1}\right)\left(\sum\frac{p_4}{p_3}\frac{p_2q_2}{\sum p_2q_2}\right)\ldots.$$

with price-updating - as required by Eurostat - we get

$$\tilde{P}^{L}_{0t}=\left(\sum\frac{p_1}{p_0}\frac{p_0q_{-1}}{\sum p_0q_{-1}}\right)\left(\sum\frac{p_2}{p_1}\frac{p_1q_0}{\sum p_1q_0}\right)\left(\sum\frac{p_3}{p_2}\frac{p_2q_1}{\sum p_2q_1}\right)\left(\sum\frac{p_4}{p_3}\frac{p_3q_2}{\sum p_3q_2}\right)\ldots.$$

which is different from $\overline{P}^{L}_{0t}$ but has (in contrast to $\breve{P}^{L}_{0t}$) an interpretation n terms of ratios of expenditures

4.1.2 (7) Different lags of prices and quantities (2)

Quantities in the weights lagging two periods: with price updating

$$\tilde{P}^{L}_{0t} = \frac{\sum p_1 q_{-1}}{\sum p_0 q_{-1}} \frac{\sum p_2 q_0}{\sum p_1 q_0} \frac{\sum p_3 q_1}{\sum p_2 q_1} \frac{\sum p_4 q_1}{\sum p_3 q_1} \frac{\sum p_5 q_2}{\sum p_4 q_2} \ldots.$$

as opposed to the genuine chain index

$$\bar{P}^{L}_{0t} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \frac{\sum p_2 q_1}{\sum p_1 q_1} \frac{\sum p_3 q_2}{\sum p_2 q_2} \frac{\sum p_4 q_3}{\sum p_3 q_3} \frac{\sum p_5 q_4}{\sum p_4 q_4} \ldots$$

The extent to which $\breve{P}^{L}_{0t}, \tilde{P}^{L}_{0t}$ and $\bar{P}^{L}_{0t}$

differ depends on changes of quantities q_{t-1}/q_{t-2} relative to changes of prices p_{t-1}/p_{t-2}.

It is recommended:

	p_{t-1}/p_{t-2}		
q_{t-1}/q_{t-2}	<< 1	≈ 1	>> 1
<< 1	c	c	a
≈ 1	b	a	b
>> 1	a	c	c

a) no price update needed

b) price up-date only

c) estimate $p_{t-1}q_{t-1}$

International Conference of Labor Statisticians ICLS 2003, § 25:

"Where the weight reference period differs significantly from the price reference period, the weights should be price updated … Where it is likely that price updated weights are less representative … this procedure may be omitted"

see also Greenlees+Williams

4.1.3 (1) Problems concerning other indices: Fixed assets, PIM and chaining

Problem addressed by Germany in the Eurostat Seminar "Introduction of Chain Indices in National Accounts" 24-25 October 2002, Luxembourg:

Calculation of capital stock using the Perpetual-Inventory-Method (PIM), when measuring volume at previous year's prices in contrast to the former **fixed price method**

What has to be done: Valuation of fixed assets at **replacement costs** of the **current** period

"The stock of fixed assets should be valued at the purchasers' prices of the current period" (ESA 6.04) "A particular item in the balance sheet should be valued as if it were being acquired on the date to which the balance sheet relates" (ESA 7.25).

Two steps. Conversion of valuations at …:

1) original **acquisition prices** → **constant prices** of a fixed year (period)
2) constant prices → **current replacement costs**

Step 1 is necessary because PIM (accumulation!) requires capital formation series (*absolute figures*, not index series) broken down by asset type in as much detail as possible and *valued in a uniform manner* at constant replacement costs, i.e. at the prices of an arbitrary fixed base year (to isolate the quantity component).

PIM = accumulation of long times series of capital formation at constant prices in absolute values

4.1.3 (2) Fixed assets: PIM and chaining (part 2)

Step two. Conversion of **constant prices → current replacement costs**

Step 2 is necessary for valuation according to ESA. This **requires**

a) price statistics (as in step 1) broken down by asset type … in order to "inflate" assets to uniformly valued at current price level, and

b) price indices that ideally measure the price trend in a way where successive periods are comparable, or in other word **"pure price comparison"** is required.

Destatis' opinion

In **traditional** fixed price base **approach** "the **price trend** between the current year and the base year for prices is **represented exactly**, whereas, the price trend in the previous year's comparison can be ascertained only to a limited extent owing to the changing weighting".

Thus: chain indices do **not** provide a pure price comparison.

and five questions →

4.1.3 (3) Fixed assets: PIM and chaining (part 3)

Destatis' questions

a) "... is it **methodologically admissible** to make the usual calculations of the capital stock … **using capital formation time series** ... which were **obtained by** the **chaining** of capital formation at previous years' prices?

b) How should the **'volume' component** be **interpreted**?" In particular: how the "**deviations** not only in the volume component, but also in the resulting replacement cost valuation" in contrast to the traditional method.*

c) How can consistency be checked in the light of the **multidimensionality of the calculations** of the consumption of fixed capital and the calculations of the fixed capital by asset types, industry, sector and market and non-market producers, if there is **no additivity** across the various dimensions?

d) Have other countries considered this problem, or do they perhaps not perceive this as a problem at all?

e) What solution is adopted in countries which already calculate the volume at previous year's prices?

* the issue was explicitly declared being not a technical one but rather a methodological (conceptual) one.

4.2 (1) USA: CPI and C-CPI of the BLS (based on Greenlees + Williams [GW])

BLS started in 2002 with two innovations to address the problem of upper level substitution bias in Lowe approach:

1. **More frequent** (biennial) **updates** of expenditure weights of the Headline **CPI-U***, formula:
low level: weighted geometric
upper: Lowe formula
Widely used for indexation because unlike C-CPI-U not subject to revisions

2. **Chained CPI**: C-CPI-U
weights: current and previous month (?)

preliminary monthly chained index; formula : "**geometric Young"** elementary indices and weights like CPI-U
Because of unavoidable lags in expenditure data subject to ***two annual revisions***

final version: **Törnquist** monthly (and monthly chained) index* designed to be a closer approximation to the COLI

Weight updates since Dec. 1998: every two years

since	weights
2002	1999 – 2000
	2001 - 2002
	2003 - 2004
2008	2005 - 2006

Data covering 8 years enabled GW to study the

- substitution behaviour
(not recorded here)
- whether weights refer to 2 years/ 1 year, and influence of
- frequency of weight revision

*all urban consumers CPI-U (≠ CPI-W)

* very much affected from volatile gasoline prices 2005/6

4.2 (2) USA: Study of Greenlees and Williams

Lowe formula
(weight base b
< price base 0)
b < 0 < t

$$P^{Lo}_{t,0,b} = \sum_i \frac{p_{it}}{p_{i0}} \frac{p_{i0}q_{ib}}{\sum p_{i0}q_{ib}} = \sum_i \frac{p_{it}}{p_{i0}} \cdot s_{i,0b} = \frac{\sum p_{it}q_{ib}}{\sum p_{i0}q_{ib}}$$

Laspeyres: b = 0; the bounding result P^L > COLI > P^P does not apply to Lowe

The importance of price updating increases with the distance between b an 0 (currently 2 years in USA)

As opposed to Lowe index the **Young index** does not involve price updating of weights

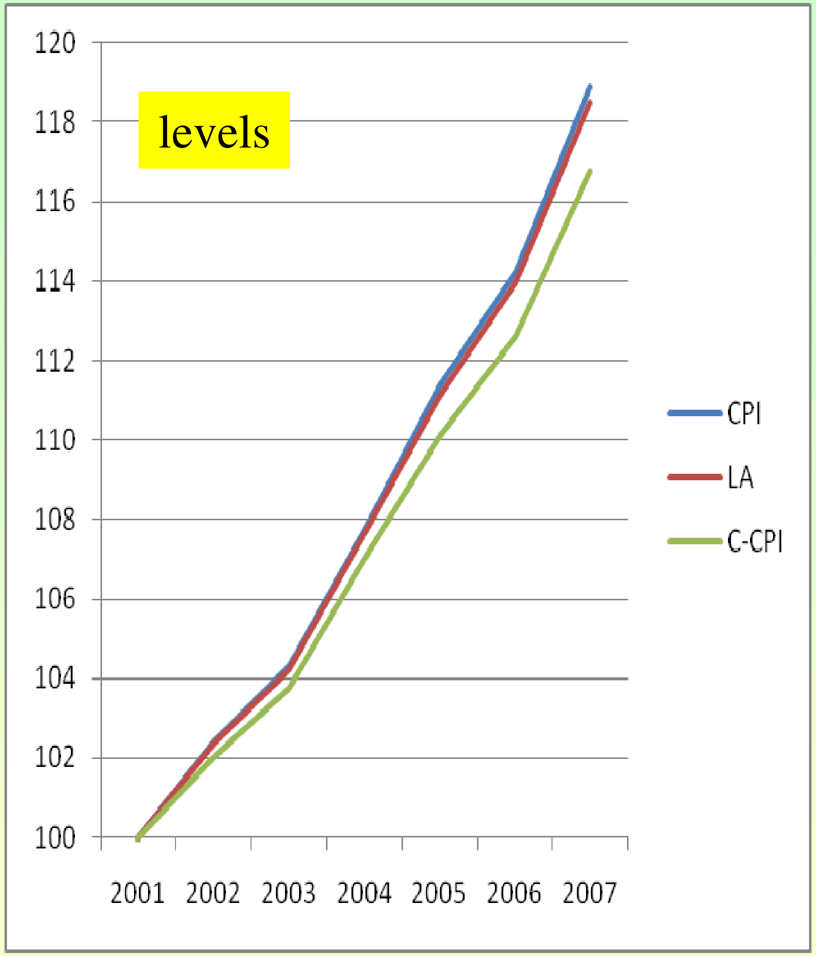
$$P^{Y}_{t,b,b} = \sum_i \frac{p_{it}}{p_{i0}} \frac{p_{ib}q_{ib}}{\sum p_{ib}q_{ib}} = \sum_i \frac{p_{it}}{p_{i0}} \cdot s_{i,bb}$$

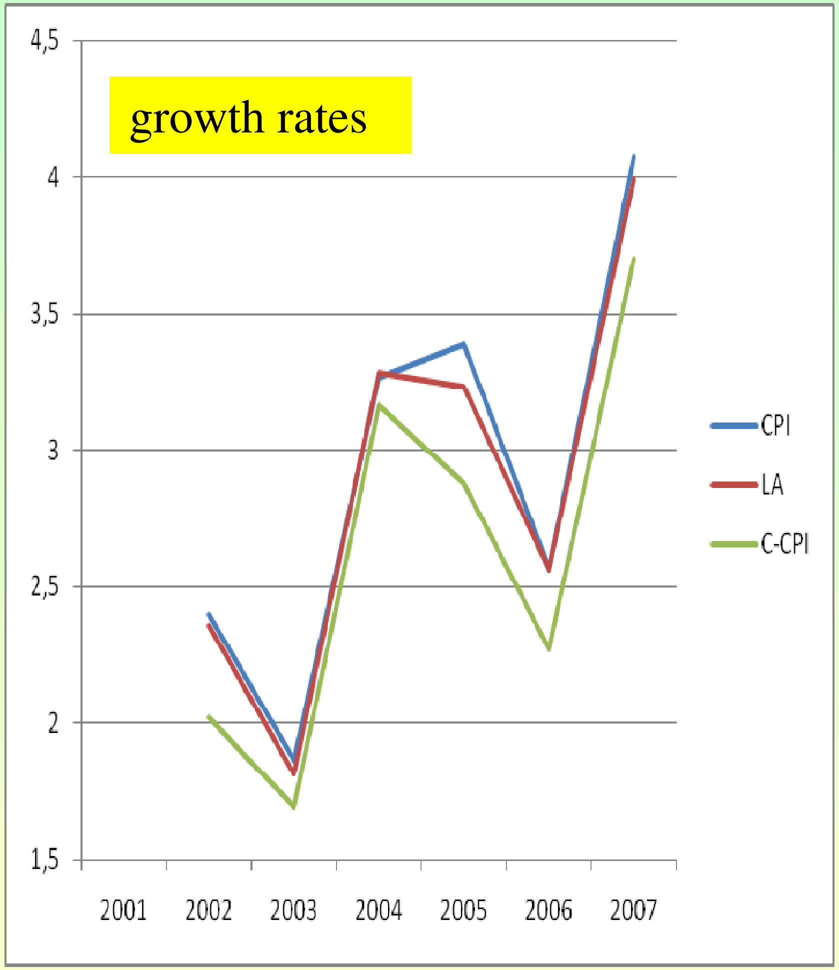
G+W studied a number of experimental indices **I(L,A,F)** by varying the parameters

I = index formula (*direct*: Young Y, Lowe L, *chained*: C)
L = **length** of weight reference period in months (e.g. 2 years, 1 year, 2 months)
A = **age** of weights (collection and processing lag) in months
F = **frequency** of updating (24 = biennial; number of months between updates)

1. "more recent (A↓) expenditure weights would typically have a downward effect"
2. "the evidence of substitution behaviour supports research on accelerated expenditure weight updates in the CPI-U" (substitution matters)
3. L↓ does not lead to more volatile indexes
4. most influential parameter: move to chain approach
5. "reducing the processing lag (A) could be as or more effective than" F

4.2 (3) Study of Greenlees and Williams: annual indices



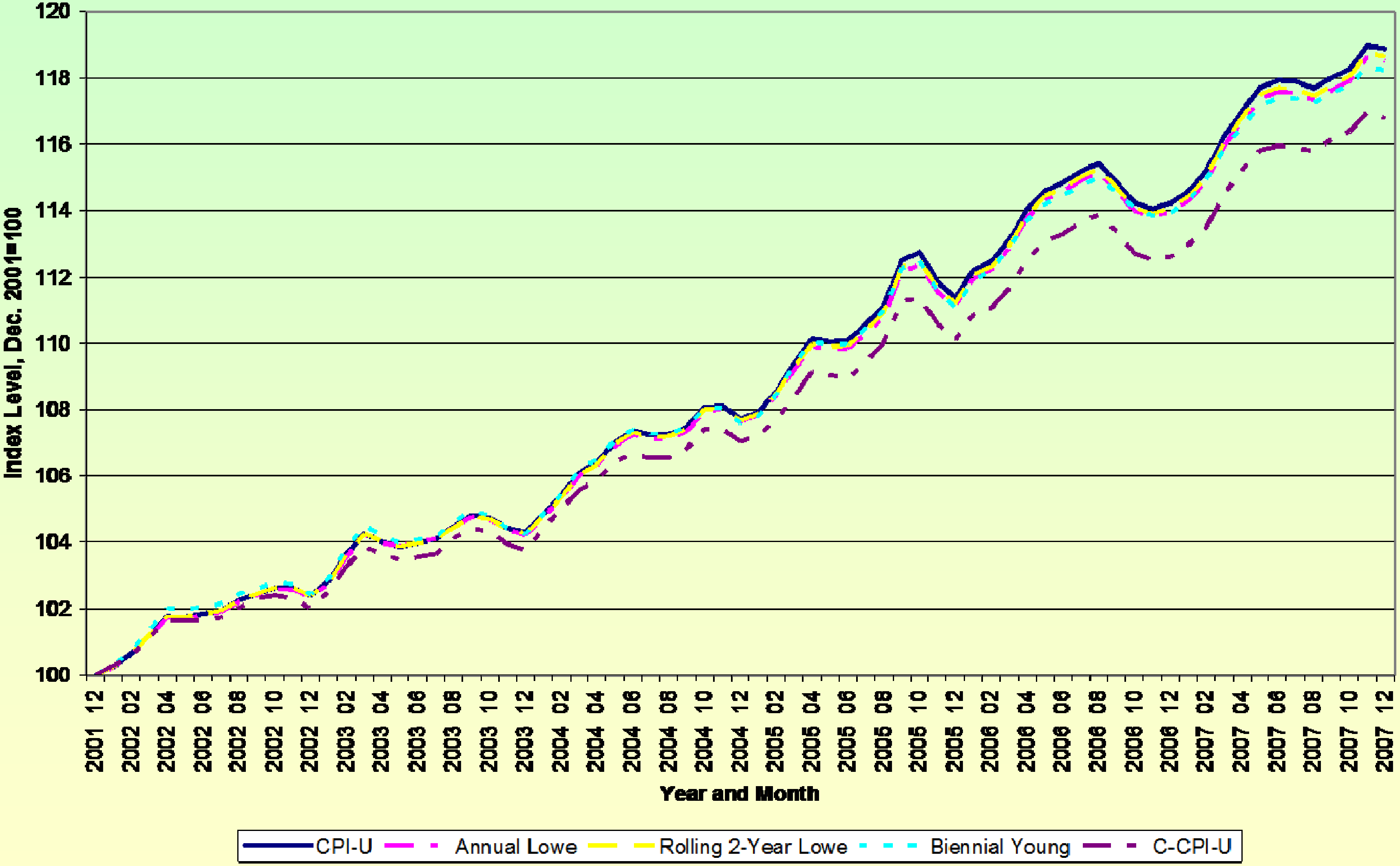


CPI = I(L,A,F) = Lowe(24,24,24)
LA = Lowe(12,18,12)
C-CPI = chained Törnquist(1,1,1)

Note: the variations of L, A, F in indices were only simulations (experiments) in retrospect. Due to data problems they cannot be performed in real time.

4.2 (4) Greenlees and Williams quarterly indices

Figure 1. Simulated Indexes With Fixed Lag Lengths



4.2 (5) Summary of Greenlees and Williams

Summary of the G+W message:

The superlative chained Törnquist C-CPI is used as the standard against which alternative formulas and operations concerning L, A, and F are judged.
Both A↓ and F↓ and in particular chaining brings an index closer to the C-CPI

"Even countries that do not accept the COLI as the conceptual objective for the CPI, however, often recognize the advantages of superlative indexes. Therefore, our overall result that more timely weights are likely to reduce the gap between a CPI and a superlative index should be of broad relevance"

For other observations (USA, Canada, Japan) concerning the relevance of frequent updating of weights see Eiglsperger + Schackis, p. 8

4.3 (1) Experiences: Does chaining matter empirically?

In former days attempts were not infrequently made to compare results gained by direct methods to those gained by chain methods.

Now, as a decision is made to use chain indices such studies would be more or less a waste of time.

The relevance of an as speedy as possible update of weights seems to be a bit exaggerated: According to the German National CPI the difference between annual inflation rates for 2006 and 2007 was only about 0.1 percentage points depending on whether weights of the year 2000 or of the year 2005 were used.

According to Schreyer, however, "chaining matters". He quotes Italian figures: growth of GDP volume (percentage) using different deflators

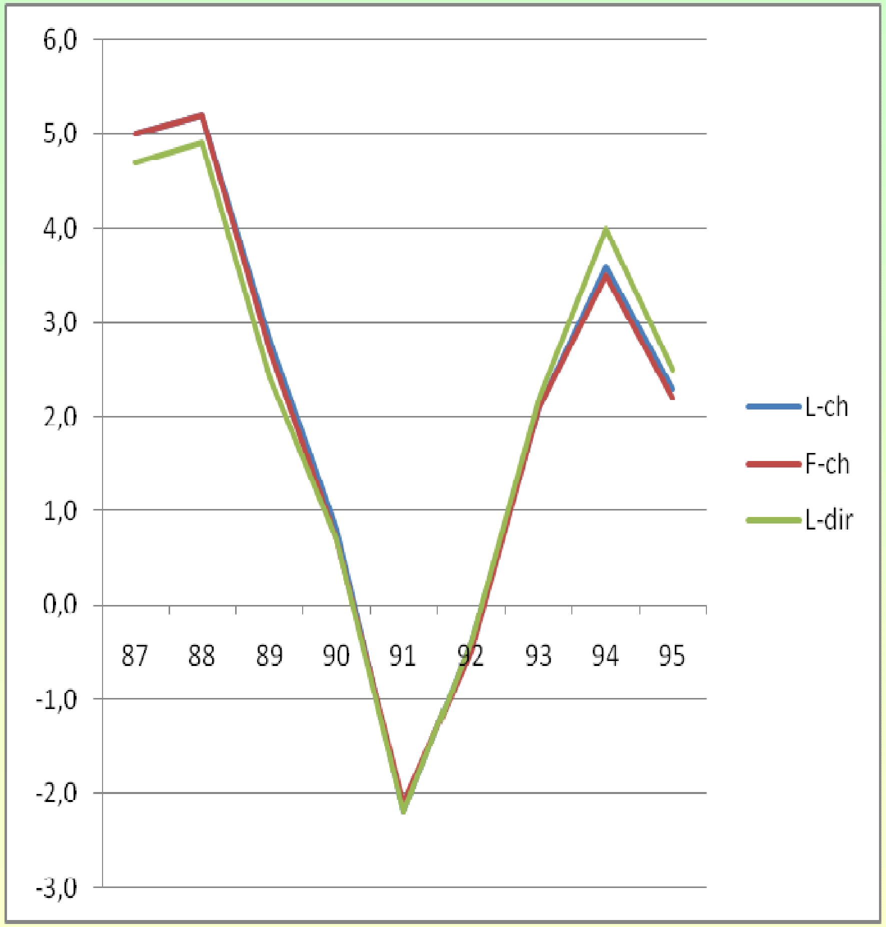
Period	P^P	$P^L ch$	$P^F ch$	$P^P ch$
1992-95	1.40	1.49	1.44	1.38
1995-98	1.64	1.59	1.58	1.56
1999-01	2.32	2.43	2.39	2.34

P^P= direct Paasche resulting in QL; $P^L ch$= Laspeyres chain etc.

4.3 (2) Experiences: Does chaining matter empirically?

Schreyer also quoted UK figures, which he thinks reflect the substitution bias. They refer to GDP "at constant prices" (%change on previous year)

	L-ch	F-ch	L-dir
87	5,0	5,0	4,7
88	5,2	5,2	4,9
89	2,8	2,7	2,4
90	0,8	0,7	0,7
91	-2,2	-2,1	-2,2
92	-0,4	-0,5	-0,4
93	2,1	2,1	2,2
94	3,6	3,5	4,0
95	2,3	2,2	2,5



Schreyer highlighted this field as indicating a high substitution bias

However, he did not say that the red fields above indicate an irrational substitution

5.1 (1) Types of aggregation and usage of the term "additivity"

the term "additivity" is used in connection with questions like

how the index reacts to changes of individual prices at time (period) 0 and/or t (as function of prices)

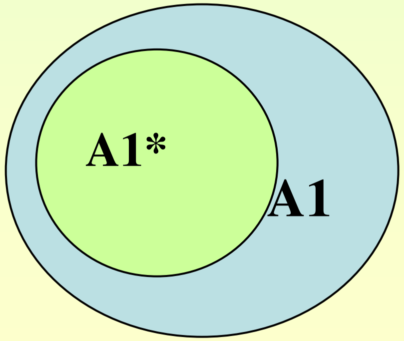
how a global index can be decomposed into sector indices, or the sector indices can be aggregated to a global index

whether the deflator provides volumes of sub-aggregates that can be summed up like values

A1*: additivity (**linearity**) of the function
(linearity of a deflator in current period prices)

A1: aggregative consistency of the index function, **ACF**
one stage and multistage compilation of the index yield the same result

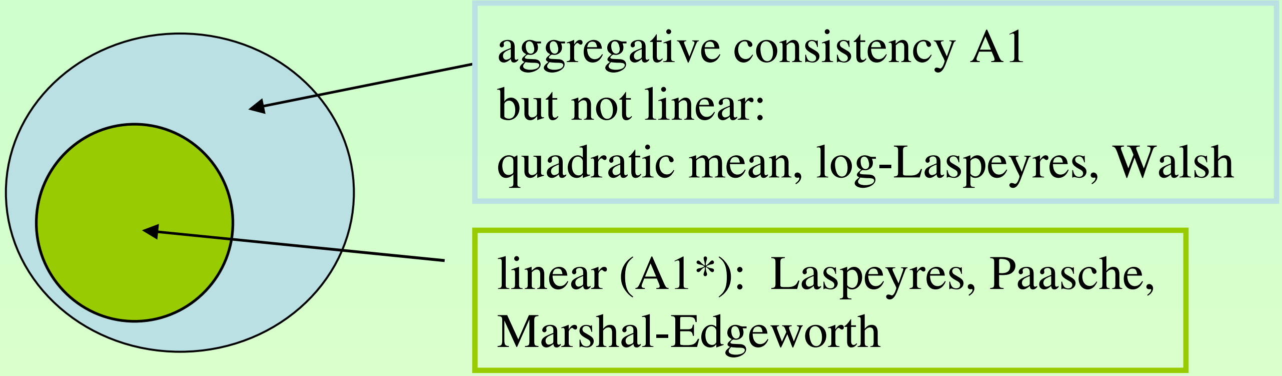
A2: structural consistency of volumes (in deflation), **SCV**
= quantity (volume) index is linear in the quantities.
SCV can only be requires using direct Paasche price indices as deflators



all other deflators (direct or chain) violate SCV

this can easily be shown ⇒

5.1 (2) Relations between aggregations concepts



aggregative consistency A1
but not linear:
quadratic mean, log-Laspeyres, Walsh

linear (A1*): Laspeyres, Paasche,
Marshal-Edgeworth

not even aggregative consistency, let alone linearity:
Fisher's "ideal" index* (of course also all sorts of chain indices)

Relevance of criterion A1 (**aggregative consistency**):

1. aggregations in 1, 2, 3 ,… steps over various aggregation levels to the all-item-index are consistent.
2. it enables users of statistics to construct their own "experimental" indices with or without certain sub-aggregates.

One might conjecture that aggregative consistency is automatically given once an index can be written as average (mean) of price relatives (mean value property). This is not true ⇒

* in 5.3 we show that Fisher's "ideal" index is anything but ideal

5.1 (3) Aggregative consistency and mean value property

An index of Drobisch , the arithmetic mean of P^L and P^P, is a mean of price relatives

$$P_{0t}^{DR} = \tfrac{1}{2}\left(P_{0t}^{L} + P_{0t}^{P}\right)$$

$$P_{0t}^{DR} = \tfrac{1}{2}\left[\left(\sum_{i=1}^{n}\frac{p_{it}}{p_{i0}}g_i\right) + \left(\sum_{i=1}^{n}\frac{p_{it}}{p_{i0}}w_i\right)\right] = \frac{1}{2}\left(P_{0t}^{L} + P_{0t}^{P}\right) = \sum_{i=1}^{n}\frac{p_{it}}{p_{i0}}\frac{g_i + w_i}{2}$$

Hence P^{DR} is clearly an arithmetic mean (unlike Fisher's index) with weights $(g_i + w_i)/2$. With n commodities grouped into two sub-indices, such that j = 1, ..., m belongs to group A, and k = m+1, ..., n to group B respectively we have

$$P_{0t}^{DR} = \tfrac{1}{2}\left[\left(\sum_{j}^{m}\frac{p_{jt}}{p_{j0}}g_j^*\right)\sum_{j}^{m}g_j + \left(\sum_{k}^{n}\frac{p_{kt}}{p_{k0}}g_k^*\right)\sum_{k}^{n}g_k + \left(\sum_{j}^{m}\frac{p_{jt}}{p_{j0}}w_j^*\right)\sum_{j}^{m}w_j + \left(\sum_{k}^{n}\frac{p_{kt}}{p_{k0}}w_k^*\right)\sum_{k1}^{n}w_k\right]$$

$g_j^* = g_j/\Sigma_j g_j$ and g_k^*, w_j^*, w_k^* correspondingly. Using

$P_{0t}^{DR} = \tfrac{1}{2}\left[P_{0t}^{LA}g_A + P_{0t}^{LB}g_B + P_{0t}^{PA}w_A + P_{0t}^{PB}w_B\right]$ $P_{0t}^{DRA} = \tfrac{1}{2}\left(P_{0t}^{LA} + P_{0t}^{PA}\right)$ P_{0t}^{DRB} analogously

$$P_{0t}^{DR} = P_{0t}^{DRA} + P_{0t}^{DRB} - \tfrac{1}{2}\left[P_{0t}^{LA}g_B + P_{0t}^{LB}g_A + P_{0t}^{PA}w_B + P_{0t}^{PB}w_A\right]$$

which is in general

not equal to $P_{0t}^{DRA}\left(\frac{g_A + w_A}{2}\right) + P_{0t}^{DRB}\left(\frac{g_B + w_B}{2}\right)$ unless $g_A = g_B = w_A = w_B = 1/2$.

5.1 (4) Structural consistency (additivity) of volumes (with direct Paasche deflation **only**)

Let V_1, V_2 , ... ,V_K denote **values** (aggregates at current prices) referring to *sub*-aggregate 1 to K, and V_T to the *total* (T) aggregate respectively, such that **by definition**

$$V_1 + V_2 + \ldots + V_K = \sum V_k = V_T \qquad k = 1, 2, \ldots, K$$

Each **volume** is defined by dividing a value by its corresponding price index (deflator), P_1, P_2, ... , P_K. To satisfy SCV the following equation has to hold for P_T, the "total deflator"

$$\frac{V_1}{P_1} + \ldots + \frac{V_K}{P_K} = \frac{V_T}{P_T}$$

Next consider value shares (or "weights") w_k to describe the fact that total value V_T is broken down into K values of sub-aggregates

$$\frac{w_1 V_T}{P_1} + \ldots + \frac{w_K V_T}{P_K} = \frac{V_T}{P_T}$$

where $w_k = \frac{V_k}{V_T}$ upon division by V_T we get

$$w_1 \frac{1}{P_1} + \ldots + w_K \frac{1}{P_K} = \frac{1}{P_T}$$

which simply means that P_T has to be a *weighted* ***harmonic*** **mean** of sectoral indices (deflators) with weights being value shares

see also double deflation (sec. **6.1 (3)**)

The only deflator price index capable of producing **structurally consistent volumes** at all levels of aggregation is the **direct Paasche** index (as this index is based on a harmonic mean of price relatives or sub-indices [sectoral deflators] respectively). Above is a "***uniqueness theorem***"

5.2.1 The notion of additivity (linearity of a quantity index)

Additivity test (Diewert)*

$$Q(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t) = \frac{\sum p_i^* q_{it}}{\sum p_i^* q_{i0}} = \frac{\mathbf{p}^{*\prime}\mathbf{q}_t}{\mathbf{p}^{*\prime}\mathbf{q}_0}$$

Q^L would be $q^* = q_0$

equivalent definition

useful consequences

theorem of Aczel and Eichhorn (1974)

Additivity in **current** period quantities

$$Q(\mathbf{q}_0,\mathbf{q}_t^*) = Q(\mathbf{q}_0,\mathbf{q}_t) + Q(\mathbf{q}_0,\mathbf{q}_t^+)$$

where $\mathbf{q}_t^* = \mathbf{q}_t + \mathbf{q}_t^+$

Additivity in **base** period quantities

$$\frac{1}{Q(\mathbf{q}_0^*,\mathbf{q}_t)} = \frac{1}{Q(\mathbf{q}_0,\mathbf{q}_t)} + \frac{1}{Q(\mathbf{q}_0^+,\mathbf{q}_t)}$$

where $\mathbf{q}_0^* = \mathbf{q}_0 + \mathbf{q}_0^+$

Neither direct Q^F nor any chain index (resulting from chain deflators) is additive

The overall percentage change in the aggregate from 0 to t can be decomposed into **contributions** of the percentage change of individual items

$$Q(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t) - 1 = \sum_i w_i \left(\frac{q_{it}}{q_{i0}} - 1 \right)$$

where $w_i = p_i^* q_{i0} \big/ \sum p_i^* q_{i0}$

Further considerations of Diewert

1) additional restrictions → Q^W (Walsh as pure Q-index)
2) w_i (additive decomposition) in the case of Q^F

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* Diewert, Lecture Notes chapter 3, p. 14 ff ** v.d.Lippe (2007), p. 193

5.2.2 (1) Theorem on linear indices (two price indices) of L. v. Bortkiewicz

The following generalized theorem of Bortkiewicz proved extremely useful

two **linear** indices
$$X_0 = \frac{\sum x_t y_0}{\sum x_0 y_0} \qquad X_t = \frac{\sum x_t y_t}{\sum x_0 y_t}$$

relatives x_t/x_0 and y_t/y_0 respectively (for example $x_t/x_0 = p_t/p_0$ and $y_t/y_0 = q_t/q_0$)* are averaged using weights $w_0 = x_0 y_0 / \Sigma x_0 y_0$ give

$$\bar{X} = X_0 \qquad \bar{Y} = \frac{\sum y_t x_0}{\sum y_0 x_0}$$

and variances
$$s_x^2 = \sum \left(\frac{x_t}{x_0} - \bar{X} \right)^2 w_0 \qquad s_y^2 = \sum \left(\frac{y_t}{y_0} - \bar{Y} \right)^2 w_0$$

and the covariance
$$s_{xy} = \sum \left(\frac{x_t}{x_0} - \bar{X} \right) \left(\frac{y_t}{y_0} - \bar{Y} \right) w_0 = \frac{\sum x_t y_t}{\sum x_0 y_0} - \bar{X} \cdot \bar{Y}$$

so that the relation between the two indices is given by
$$\frac{X_t}{X_0} = 1 + r_{xy} V_x V_y = 1 + \frac{s_{xy}}{\bar{X} \cdot \bar{Y}}$$

* this specification gives the famous relation between P^L and P^P

we made use of the theorem in **sec. 3.5.3**

5.2.2 (2) Theorem L. v. Bortkiewicz and the drift D^{PL} of the Laspeyres price index

The theorem of Bortkiewicz is particularly useful if written this way

$$s_{xy} = \sum\left(\frac{x_t}{x_0} - \overline{X}\right)\left(\frac{y_t}{y_0} - \overline{Y}\right)w_0 = \frac{\sum y_t x_0}{\sum y_0 x_0}\left(X_t - X_0\right) = \underset{Q^L}{\underset{\uparrow}{\overline{Y}}}\left(\underset{P^P}{\underset{\uparrow}{X_t}} - \underset{P^L}{\underset{\uparrow}{X_0}}\right)$$

1) If X_t and X_0 are price indices (using quantity weights y_0 or y_t respectively) then $\Sigma y_t x_0 / \Sigma y_t x_0$ must be a quantity index (Y_0 type)

2) **If $s_{xy} < 0$ then $X_0 > X_t$** (X_0 Laspeyres, X_t Paasche) **if $s_{xy} > 0$ then $X_0 < X_t$**

The theorem does not apply to products of linear indices (as eg. chain indices of the drift). We can, however, examine the **change of a drift**. Using

$$P_{02}^{L} = \frac{\sum p_1 q_0}{\sum p_0 q_0}\frac{\sum p_2 q_0}{\sum p_1 q_0} = P_{01}^{L} P_{02(1)}^{L} = g_1^0 g_2^0$$ and

$$\overline{P}_{02}^{L} = \frac{\sum p_1 q_0}{\sum p_0 q_0}\frac{\sum p_2 q_1}{\sum p_1 q_1} = P_{01}^{L} P_{12}^{L} = g_1^0 g_2^1$$

hence we have to determine the difference between →

5.2.2 (3) Theorem L. v. Bortkiewicz, drift and Hill's theory of the PLS (Paasche-Lasp.-Spread)

$$X_0 = P^L_{02(1)} = P^L_{02} / P^L_{01} = \sum p_2 q_0 / \sum p_1 q_0 = g^0_2$$ and

$$X_t = P^L_{12} = \sum p_2 q_1 / \sum p_1 q_1 = g^1_2$$

this is a result already mentioned in part I

The covariance $$s_{xy} = \sum \left(\frac{p_2}{p_1} - \overline{X} \right) \left(\frac{q_1}{q_0} - \overline{Y} \right) \frac{p_1 q_0}{\sum p_1 q_0} = \overline{Y}(X_t - X_0)$$

is responsible for the difference between X_t an X_0 and the drift $D^{PL}_{02} = \frac{\overline{P}^L_{02}}{P^L_{02}} = \frac{g^1_2}{g^0_2} = \frac{X_t}{X_0}$

negative covariance: $\overline{P}^L_{02} < P^L_{02}$ drift down (prices rise/fall in 2 in response to less/more q in 1)

positive covariance: $\overline{P}^L_{02} > P^L_{02}$ drift upwards (prices and quantities move in the same direction)

It is not so easy to study the Paasche drift D^{PP} or the Laspeyres-Paasche Gap between direct indices (γ) or chain indices because the change of D^{PP} or γ is already a matter of *more then two* indices.

5.3 (1) Fisher's ideal and superlative index far from "ideal": no aggregative consistency

Aggregation of the index formula

Direct Laspeyres and Paasche aggregate price relatives (sub-indices) $a_{0t}^{i} = \frac{p_{it}}{p_{i0}}$

using weights g_i or w_i respectively $g_i = \frac{p_{i0}q_{i0}}{\sum p_{i0}q_{i0}}$ $w_i = \frac{p_{i0}q_{it}}{\sum p_{i0}q_{it}}$

Laspeyres weights Paasche weights

Fisher's index is given by

$$(*) \quad P_{0t}^{F} = \sqrt{\left(g_1 a_{0t}^{1} + g_2 a_{0t}^{2} + \ldots + g_n a_{0t}^{n}\right)\left(w_1 a_{0t}^{1} + w_2 a_{0t}^{2} + \ldots + w_n a_{0t}^{n}\right)}$$

n goods

or $P_{0t}^{F} = \sqrt{\left(g_1 P_{0t}^{L1} + g_2 P_{0t}^{L2} + \ldots + g_K P_{0t}^{LK}\right)\left(w_1 P_{0t}^{P1} + w_2 P_{0t}^{P2} + \ldots + w_K P_{0t}^{PK}\right)}$ over K sub-aggregates

this, however, is <u>not</u> an aggregation over K subindices of Fisher k = 1, 2, …, K

$$\sqrt{\left(g_1 P_{0t}^{F1} + g_2 P_{0t}^{F2} + \ldots + g_K P_{0t}^{FK}\right)\left(w_1 P_{0t}^{F1} + w_2 P_{0t}^{F2} + \ldots + w_K P_{0t}^{FK}\right)}$$

using sectoral Fisher indices $P_{0t}^{Fk} = \sqrt{P_{0t}^{Lk} P_{0t}^{Pk}}$

P^F does not even meet the equality test $P_{0t} = f(P_{0t}^{1}, P_{0t}^{2}, \ldots, P_{0t}^{K}) = f(\lambda, \lambda, \ldots, \lambda) = \lambda$

5.3 (2) Fisher's ideal index far from "ideal": aggregation

The **equality test** requires $P_{0t} = f(P_{0t}^1, P_{0t}^2, \ldots, P_{0t}^K) = f(\lambda, \lambda, \ldots, \lambda) = \lambda$

or: if all sectoral indices P^k are equal λ, then the global index should yield = λ.

It can easily be seen that Fisher's index fails this "weak aggregation test" because two different procedures of taking an average are involved

Example

Consider two commodities and weights $g_1 = 0.6$, (consequently $g_2 = 0.6$) and $w_1 = 0.4$, ($w_2 = 0.6$) and assume sectoral indices $P^{L1} = 1.25$, $P^{P1} = 1.2$ and $P^{L2} = 2$, $P^{P2} = 0.75$

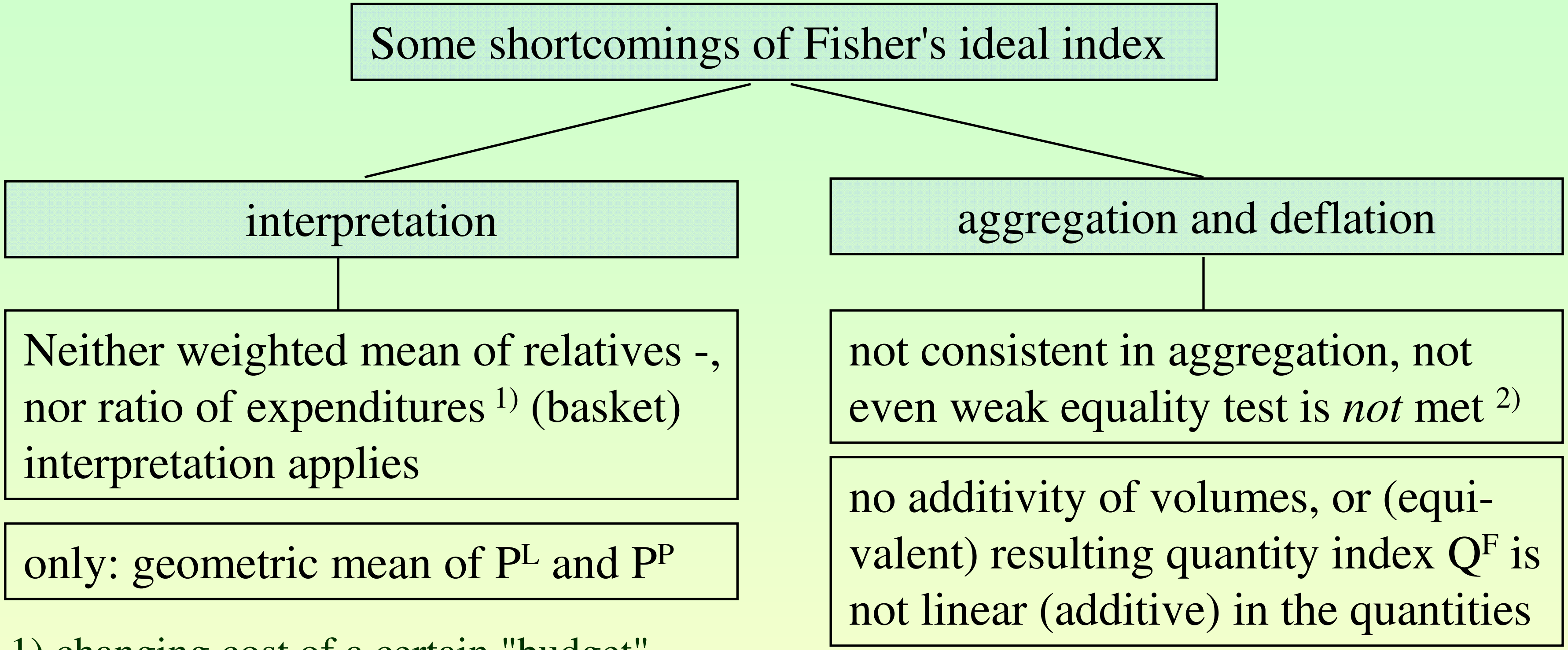
1. The sectoral Fisher indices are equal $P_{0t}^{F1} = P_{0t}^{F2} = \sqrt{1.25 \cdot 1.2} = \sqrt{2 \cdot 0.75} = \sqrt{1.5}$

2. The total Fisher index requires P^L and P^P that is $P_{0t}^F = \sqrt{\left(g_1 P_{0t}^{L1} + g_2 P_{0t}^{L2}\right)\left(w_1 P_{0t}^{P1} + w_2 P_{0t}^{P2}\right)}$

giving $\sqrt{1.55 \cdot 0.93} = \sqrt{1.4415}$ which is unequal $\sqrt{1.5}$

In 1 an unweighted geometric mean is taken,
in 2 a weighted arithmetic mean for P^L and P^P

5.3 (3) Fisher's ideal index not a good index



Some shortcomings of Fisher's ideal index

interpretation

Neither weighted mean of relatives -, nor ratio of expenditures [1] (basket) interpretation applies

only: geometric mean of P^L and P^P

aggregation and deflation

not consistent in aggregation, not even weak equality test is *not* met [2]

no additivity of volumes, or (equivalent) resulting quantity index Q^F is not linear (additive) in the quantities

1) changing cost of a certain "budget"

2) no simple function exists by which sectoral indices of P^F-type can be aggregated to a total P^F-index

Moreover: more difficulties in compiling this index (compared with P^L)

Exactly the same defects are given in the case of chain indices

6.1 (1) Task of "deflation" (a useful distinction concerning types of aggregates/deflations)

Deflation of aggregates

in volume terms "at constant prices"	**in real (income) terms** at constant purchasing power of money
Volume - approach (quantity interpretation of the result intended): isolation of the **quantity component**	**Real - income- approach** (**adjusting** income **for inflation**): estimate (adjust for) the effect of inflation
Applicable to value changes only, that can be decomposed into price- and quantity-changes, i.e. to **commodity flows** (**CF**s)	Applicable also to values that do not have price and quantity dimensions on their own, i.e. to **non-commodity flows** (**NCF**s)
Deflate aggregates A_1, A_2, ..., A_m by using **m price indices** P_i (i = 1,...,m) for the m aggregates (P_i for commodities in A_i)	Deflate aggregates A_1, A_2, ..., A_m (all or many of them NCFs) using ***one single* deflator** (for the general inflation)
Double deflation (of a difference)	Choice of **the** (general) **deflator**

6.1 (2) different methods and concepts of deflation (deflation and price level measurement)

method

- **indirect** method
  - using prices p_{i0} instead of p_{it}, i.e. by **individual re-pricing** of each aggregate
  - divide the value by an appropriate **price index** (deflator) (global re-pricing)
- **direct** method
  - extrapolation of quantities, or using an appropriate **quantity index**

Price level (inflation) measurement and deflation are

- **one** task (**SNA** recommendations)
  - **the same price index** serving both purposes; deflator of GDP, and also used to measure inflation: SNA prescribes a **chain** index of **Fisher** type
  - factor reversal test important
- **two** different tasks (traditional position)
  - requiring different methods; **two price indices**, for example P^L (CPI) **for inflation** measurement, and P^P **for deflation**
  - product test sufficient

6.1 (3) Remark on "Double deflation" with direct Paasche indices (deflators)

(t,t) denote nominal aggregates ($\Sigma q_t p_t$)
(t,0) denote real aggregates ($\Sigma q_t p_0$)

O = output, I = input, Y = value added

$$Y(t,0) = \frac{O(t,t)}{P^{P}_{0t}(O)} - \frac{I(t,t)}{P^{P}_{0t}(I)} = O(t,0) - I(t,0)$$

by definition! Rearranging gives

$$\frac{O(t,t)}{P^{P}_{0t}(O)} = \frac{Y(t,t)}{P^{imp}_{0t}(Y)} + \frac{I(t,t)}{P^{P}_{0t}(I)}$$

upon division by O(t,t) and using $i = \frac{I(t,t)}{0(t,t)}$

^{imp} = implicit

$$\boxed{\frac{1}{P^{P}_{0t}(O)} = i\frac{1}{P^{P}_{0t}(I)} + (1-i)\frac{1}{P^{imp}_{0t}(Y)}}$$

The **output deflator** $P^P(O)$ can be regarded as a weighted **harmonic mean of the input deflator** $P^P(I)$ and the **implicit value added deflator** $P^{imp}(Y)$

[both indices P(O) and P(I) of Paasche type; the weights being the quotas i and (1-i) respectively]

this result is often found counter-intuitive*: because of

input (I)	value added (Y)
Output (O)	

one would expect $P^{imp} = P^Y$ being a mean of P^O and P^I rather than P^O a mean of P^Y and P^I

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* W. Neubauer: Irreales Inlandsprodukt zu konstanten Preisen,… AStA 1974, p. 237

6.1 (4) Remark on "Double deflation" with direct Paasche deflators (2)

It is in particular possible that both indices P(O) and P(I) indicate a rise while P(Y) is showing a decline of prices. Example i = 0.7, P(O) = 1.2 and P(I) = 1.4 then

$$\frac{1}{P_{0t}^{P}(O)}=i\frac{1}{P_{0t}^{P}(I)}+(1-i)\frac{1}{P_{0t}^{imp}(Y)}$$

results in P(Y) = 0.9. Due to

$$P_{0t}^{imp}=\frac{(1-i)P_{0t}^{P}(I)P_{0t}^{P}(O)}{P_{0t}^{P}(I)-iP_{0t}^{P}(O)}$$

it is also possible that the implicit value-added-deflator is negative, indicating a negative "real" value added (VA). Example: i = 0.7, P(O) = 2.1 and P(I) = 0.8 then P(Y) = -0.7522.

What seems to be absurd is not so, however, considering the international rather than inter-temporal case. After the German reunification many East-German (GDR) deflated VAs became negative, indicating that a production might be efficient at GDR prices , but would be no longer be profitable at West German prices.

Direct Fisher price index as deflator makes things more complicated

$$\frac{1}{P_{0t}^{P}(O)}=i\frac{1}{P_{0t}^{P}(I)}R_{1}+(1-i)\frac{1}{P_{0t}^{imp(F)}(Y)}R_{2}$$

where $R_1^2 = R_2^2\left[P_{0t}^{P}(I)/P_{0t}^{L}(I)\right]$

and $R_2^2 = P_{0t}^{P}(O)/P_{0t}^{L}(O)$

Paasche/
Laspeyres
ratio

The sum of the weights $iR_1 + (1-i)R_2 \neq 1$, and R_2 becomes important later ⇒ **6.4.1** (slide 52)

6.2.1 (1) Different notions of "volume"

Sequence of volumes (monetary terms in €)

Should they be published in addition to indices? **How should they be called?** ⇒

	t = 1	t = 2	t = 3	general
traditional method (Paasche direct)	$\sum q_1 p_0$	$\sum q_2 p_0$	$\sum q_3 p_0$	$\sum q_t p_0$
at **previous year** prices	$\sum q_1 p_0$	$\sum q_2 p_1$	$\sum q_3 p_2$	$\sum q_t p_{t-1}$
new method "**chained volumes**"(?) updated using chain indices)	$\sum q_1 p_0$	$\sum q_1 p_0 \frac{\sum q_2 p_1}{\sum q_1 p_1}$	$\sum q_1 p_0 \frac{\sum q_2 p_1}{\sum q_1 p_1} \frac{\sum q_3 p_2}{\sum q_2 p_2}$	

here we rightly speak of "at constant prices" (of the base period)

here: a chain Paasche index (for other chain index deflators see **6.4.3**)

general

$$\sum q_1 p_0 \frac{\sum q_2 p_1}{\sum q_1 p_1} \frac{\sum q_3 p_2}{\sum q_2 p_2} \frac{\sum q_4 p_3}{\sum q_3 p_3} \cdots \frac{\sum q_t p_{t-1}}{\sum q_{t-1} p_{t-1}}$$

factors resemble those of the AO method (see part III)

6.2.1 (2) Official terms for volumes derived from chain-index-deflation* (in 2007)

country	base**	terminology "for volumes" (2004)
Belgium	2000	in chained 2004 euros
Finland		at reference year 2000 prices
France		chained prices base 2000
Greece		constant prices of the previous year
Ireland	2005	constant market prices (chain linked annually and referenced to year 2005)
Italy		chain-linked volumes 2000 = 100
Netherlands	2000	prices of 2000
Portugal		- at prices of the previous year - chain linked volume data (reference) year = 2000
Denmark		2000 price level chain figures
Sweden		constant prices reference year 2000, chain linked series

contradiction in terms

* according to Leifer/Tennagels ** or reference) period

6.2.2 (1) Volumes at previous year prices and (decomposition of) their growth rates

A thought experiment of Tödter (2005): assume two good with constant changes of prices and quantities over time: $p_{10} = p_{20} = p_0$ and $q_{10} = q_{20} = q_0$. Furthermore
$p_{1t} = p_0(1+\pi)^t$, $q_{1t} = q_0(1-\pi)^t$ and $p_{2t} = p_0(1-\pi)^t$, $q_{1t} = q_0(1+\pi)^t$

Tödter: Volumes **at prices of the previous year** (Vorjahrespreismethode) remain **constant** (growth rate = 0 for all t) $Q_1 = p_0 q_0\big((1-\pi)+(1+\pi)\big) = 2p_0 q_0 = Q_0$

wrong: they are constantly declining

(though quantities are rising)

$$Q_2 = 2p_0 q_0 (1-\pi)(1+\pi) = 2p_0 q_0 (1-\pi^2)$$

$$Q_3 = 2p_0 q_0 (1-\pi^2)^2$$

$$Q_4 = 2p_0 q_0 (1-\pi^2)^3$$

$$\longrightarrow \frac{Q_t}{Q_{-1}} = 1-\pi^2 = \text{const} < 1$$

Tödter: Volumes **at constant prices of the base year** (Festpreismethode) are constantly **rising** by

this is **correct**

$$Q_1^* = p_0 q_0\big((1-\pi)+(1+\pi)\big) = 2p_0 q_0 \qquad Q_2^* = p_0 q_0\big((1-\pi)^2+(1+\pi)^2\big)$$

$$\sum q_1 = q_0\big((1-\pi)+(1+\pi)\big) = 2q_0 \qquad \sum q_2 = q_0\big((1-\pi)^2+(1+\pi)^2\big)$$

moreover: they are rising **at the same rate as total quantities Σq_t**

$$\frac{Q_t^*}{Q_{t-1}^*} = \frac{\sum q_t}{\sum q_{t-1}} = \frac{(1+\pi)^t + (1-\pi)^t}{(1+\pi)^{t-1} + (1-\pi)^{t-1}}$$

6.2.2 (2) Volumes at previous year prices: Tödter's formulas $\pi = 0.1$ $(1-\pi^2 = 0.99)$

	at prices of preceding period		constant prices of base period	
t	volume Q	growth rate (%)	volume Q*	growth rate (%)
1	2	0	2	0
2	1.98	-1	2.02	+1
3	1.9602	-1	2.06	+1.98
4	1.9406	-1	2.1202	+2.92
5	1.9212	-1	2.2010	+3.81
6	1.8830	-1	2.3030	+4.63

volume at const. prices of t = 0

$$\frac{Q_t^*}{Q_{t-1}^*} = (1+\pi)\omega_{t-1} + (1-\pi)(1-\omega_{t-1})$$

growth factor of Q* as weighted average of (1+π) and (1-π).

$$\omega_{t-1} = \frac{(1+\pi)^{t-1}}{(1+\pi)^{t-1} + (1-\pi)^{t-1}} \quad (\pi > 0)$$

since $\lim_{t\to\infty}\omega_{t-1} = 1$

the growth rate tends to π (+10%)

the **value** $\Sigma p_t q_t$ (nominal aggregate) is changing as follows ($V_0 = 2p_0q_0$)

$V_1 = \Sigma p_1q_1 = 2p_0q_0(1-\pi^2) = (1-\pi^2)V_0$

$V_2 = \Sigma p_2q_2 = 2p_0q_0(1-\pi^2)^2 = (1-\pi^2)V_1$ etc.

volumes Q are obviously ***not*** constant

constant prices: volumes develop like quantities (volumes Q* rising while values [and implicit price index] are decreasing)

volumes Q* at **prices of preceding period**: volumes develop like values; implicit price index = 1

6.3 (1) Criteria for good deflation (in volume terms)

Aim: **"volume" as a proxy of "total quantity"** (quantities cannot be added, so we volumes as a proxy)

To find criteria (quasi "axioms") consider the following simple situations

Prices	Quantities change at	
	(1) the same* rate ω	(2) different rates
(1) same* rate λ	case 11	case 12
(2) different rates	case 21	case 22

* the case of constant prices/quantities is the special case of $\lambda = 1$, or $\omega = 1$ respectively

Case 11 is clearly the simplest situation: one would expect volume to change at the rate ω. **Volumes** should be **proportional in the quantities**.

We will see what happens in the direct and chain deflator case by means of an example →

6.3 (2) Criteria for deflation: case 11 (prices and quantities change at the same rate)

Deflation using **direct** Fisher price indices yield non-additive volumes. **Chain** Fisher price indices as deflators are even worse: **in addition to non-additivity** also **proportionality** (and thus identity) is **violated**

Assume that prices of two goods, A and B are rising uniformly by 50% from 0 to 3, and quantities remain constant such that the value index **all direct price indices** (P, L, F) amount to **1.5**

	period 0		period 1		period 2		period 3	
good	p	q	p	q	p	q	p	q
A	30	5	40	3	50	2	45	5
B	10	15	5	20	10	13	15	15

P_{03}^F as **all** other **direct indices** yields **1.5**

$\Sigma p_0q_0 = \Sigma p_0q_t = 300$ so the volume should be 300 and the value $\Sigma p_tq_t = \Sigma 1.5p_0q_0 = 450$

chain index deflators and their volumes

$$\overline{P}_{03}^F = 1.5\sqrt{\frac{\sum p_1q_0 \sum p_2q_1 \sum p_0q_2}{\sum p_0q_1 \sum p_1q_2 \sum p_2q_0}} = 1.5\sqrt{1.087} = 1.564$$

according to the chain index prices rose by more than 50%

and therefore volume: 450/1.564=287.71 instead of 300

6.3 (3) Criteria for deflation: cases 11+12 (prices change at the same rate)

This defective deflation is caused by the fact that chain price indices fail proportionality (in prices) so chain deflators fail proportionality in the quantities

Though prices changed unanimously by + 50% and volume remained constant 300 we have

$$\overline{P}_{03}^{P} = 1.5 \frac{\sum p_1 q_0 \sum p_2 q_2 \sum p_0 q_0}{\sum p_0 q_1 \sum p_1 q_2 \sum p_2 q_0} = 1.354 \longrightarrow \quad \text{volume: } 450/1.354 = 332.35$$

$$\overline{P}_{03}^{L} = 1.807 \Rightarrow 249.03 \qquad \overline{P}_{03}^{ME} = 1.5554 \Rightarrow 289.32$$

case 12: again $p_{it}/p_{i0} = \lambda \ \forall i$ but q_{it}/q_{i0} may be different

To arrive at a meaningful "volume" it appears reasonable to simply divide $\Sigma p_t q_t$ by λ (the uniform inflation rate) which gives $\Sigma p_0 q_t$ (acceptable also any weighted sum of quantities $\Sigma \alpha q_t$ so that $\Sigma \alpha q_t / \Sigma \alpha q_0$ represents the volume change).*

	period 0		period 1		period 2		period 3	
	p	q					p	q
A	30	5					45	12
B	10	15					15	18

*It is not reasonable to require a change $\Sigma q_t / \Sigma q_0$

in the case of this example direct index deflators are equal $P_{03}^{P} = P_{03}^{L} = P_{03}^{F} = 1.5$

and yield the same volume 810/1.5 = 450

6.3 (4) Criteria for deflation: cases 12 (prices same rate) and 21 (quantities same rate)

Chain index deflators will not necessarily result in P = 1.5 giving a volume of 450

	period 0		period 1		period 2		period 3	
	p	q	p	q	p	q	p	q
A	30	5	40	3	50	2	45	12
B	10	15	5	20	10	13	15	18

value $\Sigma p_t q_t = 810$

Their result depends on the "path" (intermediate periods, here the white fields)

$$\overline{P}_{03}^{P} = 1.3178$$

$$\overline{P}_{03}^{L} = 1.8071$$

$$\overline{P}_{03}^{F} = 1.5431$$

generating volumes between 446.24 and 614.67

case 21: now $q_{it}/q_{i0} = \omega \ \forall i$ but p_{it}/p_{i0} may be different

When for all quantities holds $q_{it} = \omega q_{i0}$, the value at t is in actual fact simply $\Sigma p_t q_t = \omega \Sigma p_t q_0$ and – in line with the principle of pure quantity comparison – the measure of volume change (relative to the volume at t = 0) should be ω as all quantities changed by the same rate ω.

Which volume at time t (vol_t) is implied using a deflator P given that by definition $\Sigma p_t q_t / P = vol_t$?

$P = P^P \rightarrow vol_t = \omega \Sigma p_0 q_0 = \Sigma p_0 q_t$

$P = P^F$ yields the same result $vol_t = \Sigma p_0 q_t$

6.3 (5) Criteria for deflation: cases 21 and 22

While we get with a **direct** Paasche or direct Fisher *the same reasonable result* , viz.

$\text{vol}_t = \omega\sum p_0q_0$ this will no longer hold once **chain indices** are used **as deflators**.

To make it simple assume only two links and $\Sigma p_2q_2 = \omega\Sigma p_2q_{0,}$. The volumes then are

$$\frac{\sum p_2q_2}{\overline{P}^P_{02}} = \frac{\omega\sum p_2q_0}{\overline{P}^P_{02}} \Rightarrow \text{vol}_2 = \omega\frac{\sum p_0q_1\sum p_1q_0}{\sum p_1q_1} \neq \omega\sum p_0q_0$$

and with a chain Fisher index

$$\frac{\sum p_2q_2}{\overline{P}^F_{02}} \Rightarrow \text{vol}_2 = \omega\sqrt{\frac{\sum p_2q_0\sum p_0q_1\sum p_0q_0}{\sum p_2q_1}} \neq \omega\sum p_0q_0$$

To get the same result $\text{vol}_2 = \omega\sum p_0q_0$ as with direct indices requires

$$\overline{P}^P_{03} \to \frac{\sum p_1q_1}{\sum p_0q_1} = \frac{\sum p_1q_0}{\sum p_0q_0} \text{ or } P^L_{01} = P^P_{01} \qquad \overline{P}^F_{03} \to \frac{\sum p_2q_1}{\sum p_1q_1} = \frac{\sum p_2q_0}{\sum p_0q_0}$$

case 22: both prices as well as quantities may change at **different** rates ω_i (quantities) and λ_i (prices)

It does not seem to be easy to find criteria for a reasonable deflation in this situation

6.3 (6) Criteria for deflation

Reasonable though most restrictive is in this case **pure quantity comparison**, or equivalent, linearity in the quantities:

This requires the **movement of volumes** to be **reflective of changes in the quantities** irrespective of how prices changed (uniform or non-uniform). This is also equivalent to **additivity of the volumes** gained by such a deflation.

case: uniform change as regards	deflation should fulfil	direct indices	chain indices
11 both prices *and* quantities	proportionality in the quantities q_t	all pass this test	all fail this test
12 prices only	volume change $\Sigma\alpha q_t/\Sigma\alpha q_0$*		
21 quantities only	volume change should equal ω		
22 neither prices nor quantities	resulting volume index linear in quantities	only P^P as deflator	

To sum up:

In addition to non-additivity (applies also to P^F) chain indices may not respond correctly to some simple scenarios

* this is the same as a linear quantity index, normally very restrictive however, easily met in such a situation

6.3 (7) Criteria: Why proportionality or even linearity in the quantities is desirable?

No chain-index deflator is able to ensure proportionality in the quantities let alone additivity (linearity) in the quantities. So why this is a serious defect?

Given some base period values $V^B = \Sigma p_0 q_0$ for any $k = 1,...,K$, as for example k=1 private consumption, and k=2 investment it might be desirable to "update" these aggregates using suitable quantity indices Q_k, such that

$$V_1^B Q_1 + ... + V_K^B Q_K = (V_1^B + V_K^B) Q_T$$

It may e.g. be an option (or superior method) to extrapolate quantities using an appropriate quantity index (= direct method of deflation)*

The only total-aggregate (Q_T) quantity index permitting this type of consistent "updating" of base period (sub-aggregate) volumes to current period volumes needs to be an *arithmetic* mean of Q_1, Q_2, ... with weights g_k, hence a *Laspeyres quantity index* as the counterpart to the harmonic mean (Paasche) of prices:

The harmonic mean in P corresponds to an arithmetic mean in Q, such that we get the pair P^P, Q^L. In our view this is more reasonable than to seek for factor reversibility.

* The Handbook on volume measurement considers this method (to use indicators of quantity) in particular in the case of non-market production (government, education, administration, health etc.).

6.3 (8) However: a justified critique of traditional (direct Paasche) deflation

Volumes and growth rates of volumes will differ depending on which year is chosen as price basis

History has to be re-written whenever we switch to a new base?

However, it is clear that

$\Sigma p_0q_6, \Sigma p_0q_7, \Sigma p_0q_8, \Sigma p_0q_9$ …. and $\Sigma p_5q_6, \Sigma p_5q_7, \Sigma p_5q_8, \Sigma p_5q_9, \ldots$

will in general differ (to expect otherwise would imply transitivity)

Moreover, it is clear that $\sum p_0q_1, \sum p_0q_2, \sum p_0q_3, \ldots$ is a series at constant prices of period 0. But does this apply also to

$$\sqrt{\sum p_0q_0 \sum p_1q_1 \frac{\sum p_0q_1}{\sum p_1q_0}} \qquad \sqrt{\sum p_0q_0 \sum p_2q_2 \frac{\sum p_0q_1 \sum p_1q_2}{\sum p_1q_0 \sum p_2q_1}}$$

the series we get with
chained Fisher deflator

$$\sqrt{\sum p_0q_0 \sum p_3q_3 \frac{\sum p_0q_1 \sum p_1q_2 \sum p_2q_3}{\sum p_1q_0 \sum p_2q_1 \sum p_3q_2}}$$

6.4 Direct and chain price indices as deflators

Section **6.3** has shown that **all chain** price indices as **deflators** yield volumes that are both

By contrast volumes gained by **direct deflators**

(1) not proportional in the quantities → • meet proportionality in all cases

(2) not additive → • fail additivity in all cases except direct Paasche

In **6.4.1** we examine the relation between direct Fisher and direct Paasche volumes
In **6.4.2** between direct Paasche and chain Paasche

non-additivity as such (in the case of chain Paasche) is well known, however, we study a process of "**eternal recurrence**" where the same price-quantity situation repeats itself after $\Delta = 6$ periods so that

volumes
price indices
deviations due to non-additivity

ought to be the same in 0 and 7, 1 and 8, …

In **6.4.3** we examine series of volumes resulting from various methods of deflation (using direct or chain indices). It turns out that the series of chain-method-volumes are complicated and thus difficult to interpret.

6.4.1 Direct Paasche and direct Fisher price indices as deflators

Fisher volume (quantity) indices Q^F – resulting from Fisher deflation – differ from the respective Laspeyres indices Q^L as follows:

$$\frac{Q^F_{0t}}{Q^L_{0t}} = \sqrt{\frac{Q^P_{0t}}{Q^L_{0t}}} = \sqrt{\frac{P^P_{0t}}{P^L_{0t}}}$$

so $Q^F_{0t} < Q^L_{0t} \text{ if } P^P_{0t} < P^L_{0t}$ and the Bortkiewicz relation holds (covariance between price and quantity relatives, denoted by a and b)

$$Q^F_{0t} = Q^L_{0t}\sqrt{1 + r_{ab}V_aV_b}$$

Assume an aggregate S as the sum of sub-aggregates A and B. The value ($\Sigma p_t q_t$) of S then is $V_S = V_A + V_B$.

Deflation of these values **using direct Paasche** price indices for A, B and S gives

$$\overline{V}_S = \frac{V_S}{P^{PS}} = \frac{V_A}{P^{PA}} + \frac{V_B}{P^{PB}} = \overline{V}_A + \overline{V}_B$$

This condition of additivity holds as the Paasche deflator is a harmonic mean

$$\left(P^{PS}\right)^{-1} = \frac{\frac{1}{P^{PA}}V_A + \frac{1}{P^{PB}}V_B}{V_A + V_B}$$

The equivalent equation when **direct Fisher** indices are used as deflators

$$\left(\overline{V}_A + \overline{V}_B\right)\sqrt{\frac{w_A P^{PA} + w_B P^{PB}}{g_A P^{LA} + g_B P^{LB}}} = \overline{V}_A\sqrt{\frac{P^{PA}}{P^{LA}}} + \overline{V}_B\sqrt{\frac{P^{PB}}{P^{LB}}}$$

where use is made of weights like in **5.3 (1)**

$$g_A = \frac{p_{A0}q_{A0}}{\sum p_{i0}q_{i0}}$$

$$w_A = \frac{p_{A0}q_{At}}{\sum p_{i0}q_{it}} = \frac{\overline{V}_A}{\overline{V}_A + \overline{V}_B}$$

Additivity is valid only in very special cases, e.g. if $P^{PA} = P^{PB} = P^{PS}$ or $P^{LA} = P^{LB} = P^{LS}$. The total Fisher index P^{FS} (the √ on the LHS) is *not* a harmonic mean

6.4.2 (1) Example: Direct Paasche (= at constant prices) and additivity

t	pa1	qa1	pa2	qa2	pb1	qb1	pb2	qb2	
0	30	70	50	30	90	20	120	130	
1	45	84	75	48	135	36	180	24	
2	54	100	60	77	121	65	252	19	
3	65	110	70	80	130	60	230	25	
4	70	90	68	85	120	68	210	30	
5	75	120	80	70	135	45	220	80	
6	50	80	70	50	100	30	170	100	
7	30	70	50	30	90	20	120	130	same as t = 0
8	45	84	75	48	135	36	180	24	same as t = 1
9	54	100	60	77	121	65	252	19	same as t = 2

commodities
a1, a2 of A
b1, b2 of B

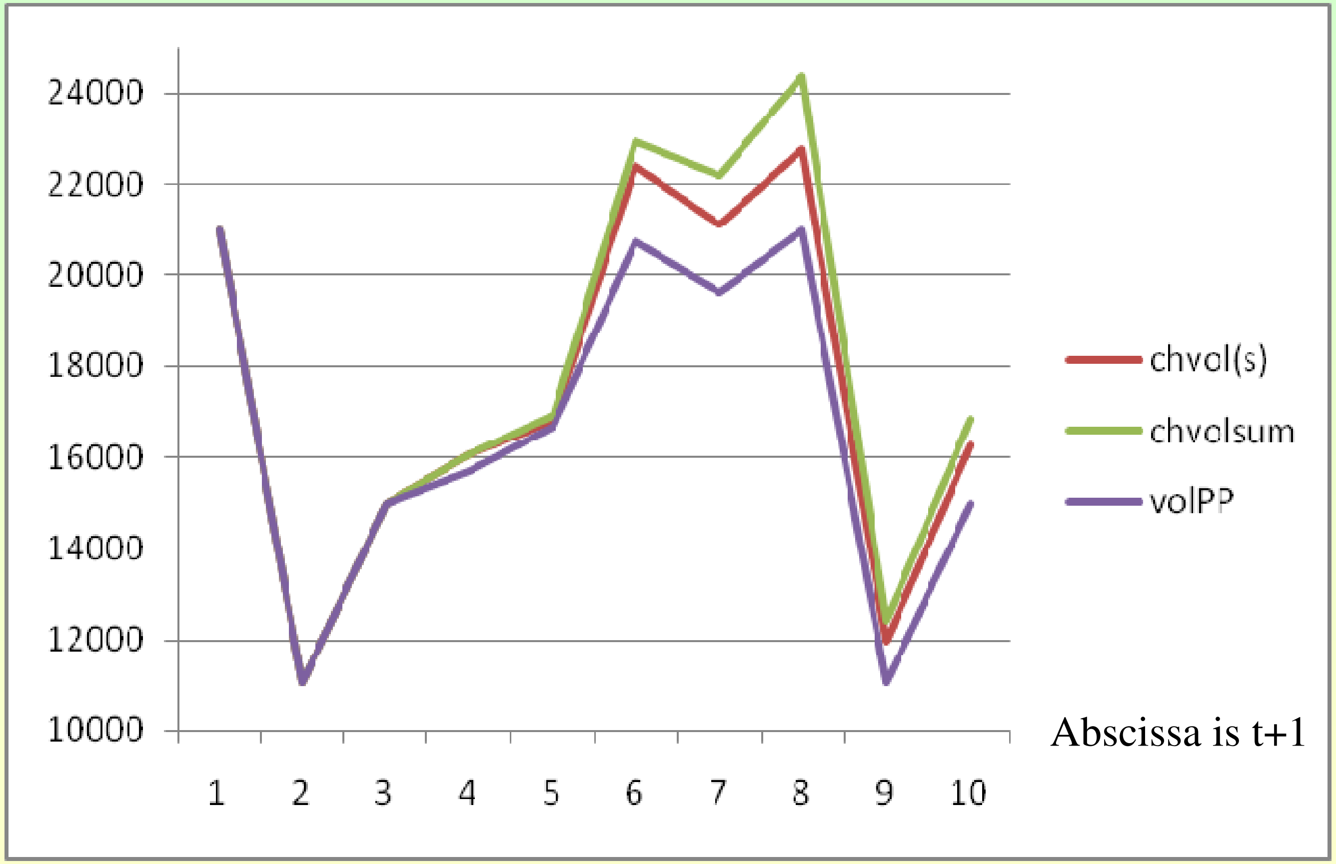
p = price
q = quantity

For t = 0 and t = 1 is this ex. 5.2.1 of v.d.Lippe (2007),

6.4.2 (2) Total volumes, Non-additivity of chain-deflator volumes

Volumes derived from

1. Paasche chain index (total aggregate deflated = **chvol(s)** and sum of the partial (aggregates A and B) volumes = **chvolsum**
2. direct Paasche index = **volPP**



Abscissa is t+1

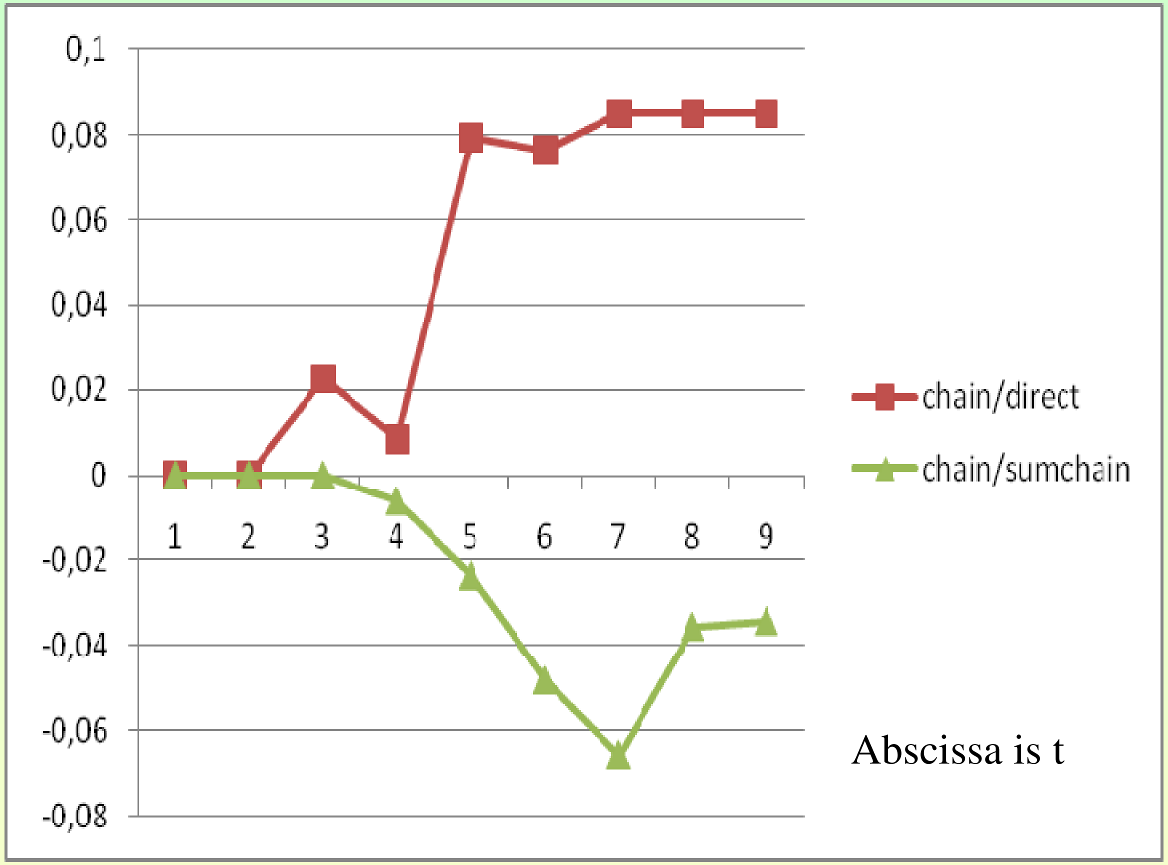
Volumes should be identical in periods

0 and 7
8 and 1
9 and 2

Higher chained volumes in periods 5 to 7 because chained deflator is smaller than direct Paasche deflator

Divergence of **chvol(s)** and **chvolsum** because **chained volumes are not additive**

6.4.2 (3) Discrepancies between volumes, Non-additivity of chain-deflator volumes



chained volumes (chain) are up to 8% higher than const. prices volumes (direct)

chained volume of sum (chain) smaller than sum of chained volumes (sumchain)

t	D%
0	0,00
1	0,00
2	0,00
3	0,00
4	-0,62
5	-2,36
6	-4,83
7	-6,57*
8	-3,56
9	-3,43

Discrepancy due to non-additivity is not substantial

Example: in period 7 chain-volume of total aggregate is by 6.57 % lower than sum of the chain volumes of aggregates A and B

6.4.2 (4) Direct Paasche and chained Paasche as deflators (**total aggregate***)

Abscissa is t+1

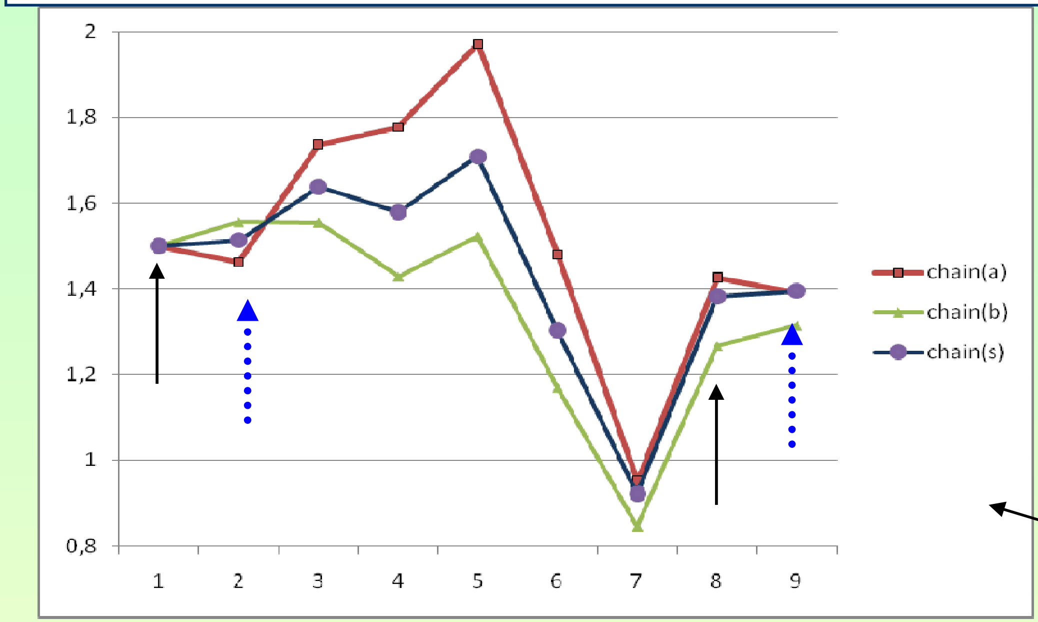
t	PPchain(s)	PPdir(s)
0	1,000	1,000 ←
1	1,500	1,500 ←
2	1,514	1,514 ←
3	1,638	1,675
4	1,579	1,592
5	1,709	1,845
6	1,304	1,403
7	0,922	1,000 ←
8	1,383	1,500 ←
9	1,395	1,514 ←

Direct Paasche is indeed the same in **0 and 7**, **1 and 8**, and in **2 and 9**

As a rule chain index is lower than **direct index**

* a closer look at the components (sub-aggregates) →

6.4.2 (5) Price indices (deflators) for the components (**subaggregates**) A and B

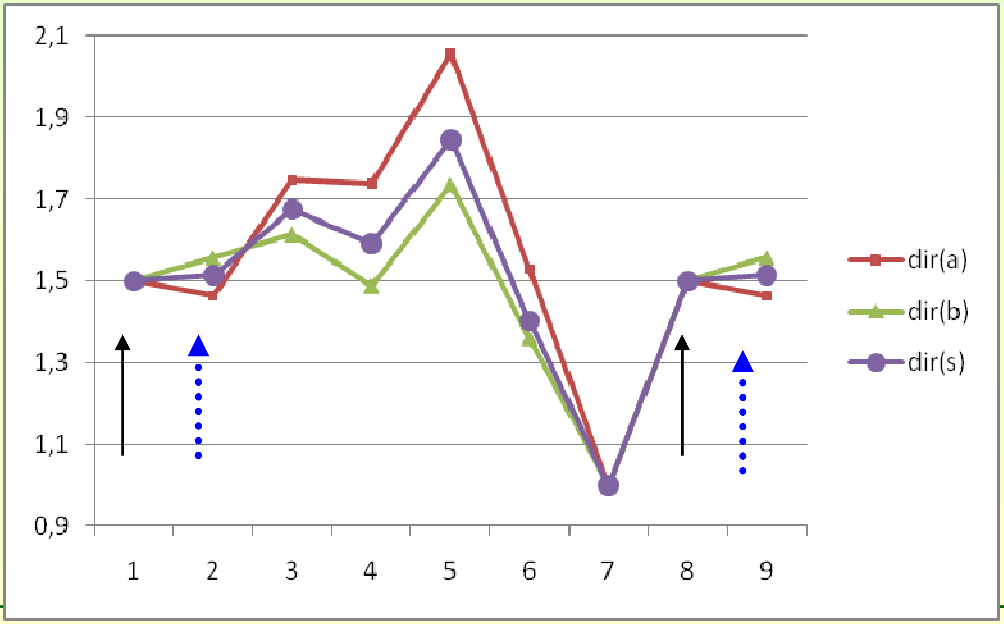


chained Paasche indices

(s = total aggregate [a+b])

Price indices do not repeat themselves after Δ periods

Mean value property



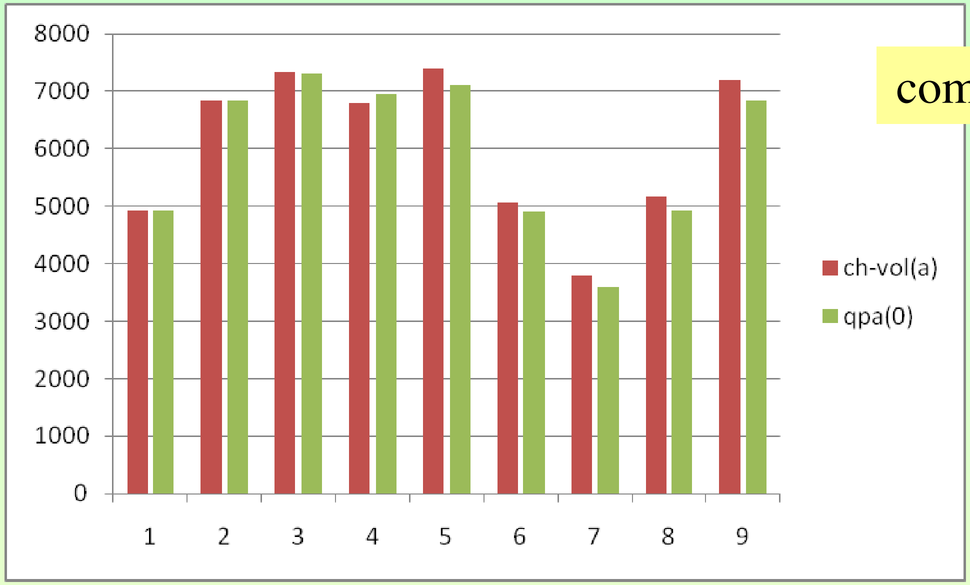
direct Paasche indices

Price indices repeat themselves after Δ periods

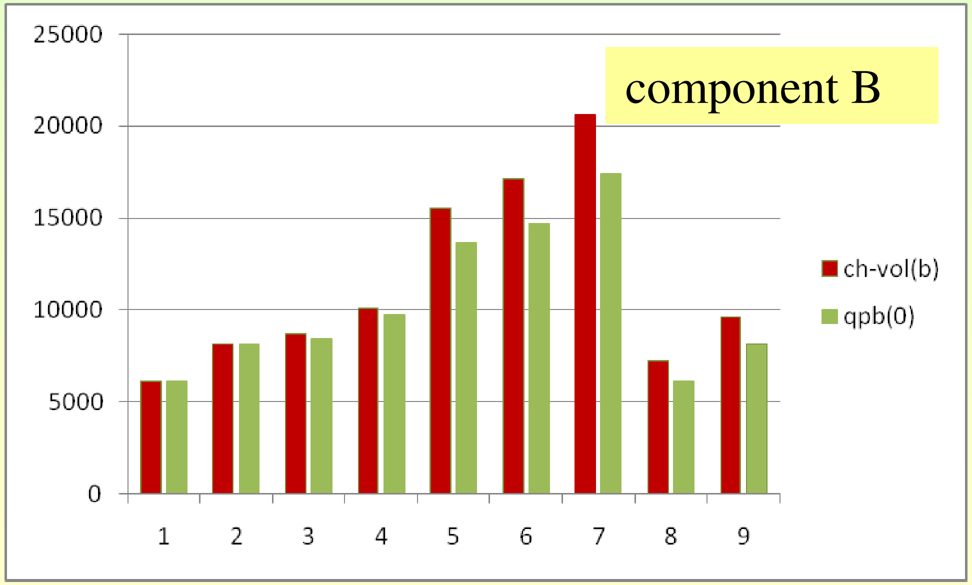
6.4.2 (6) Volumes (chained and at constant p_0 prices)

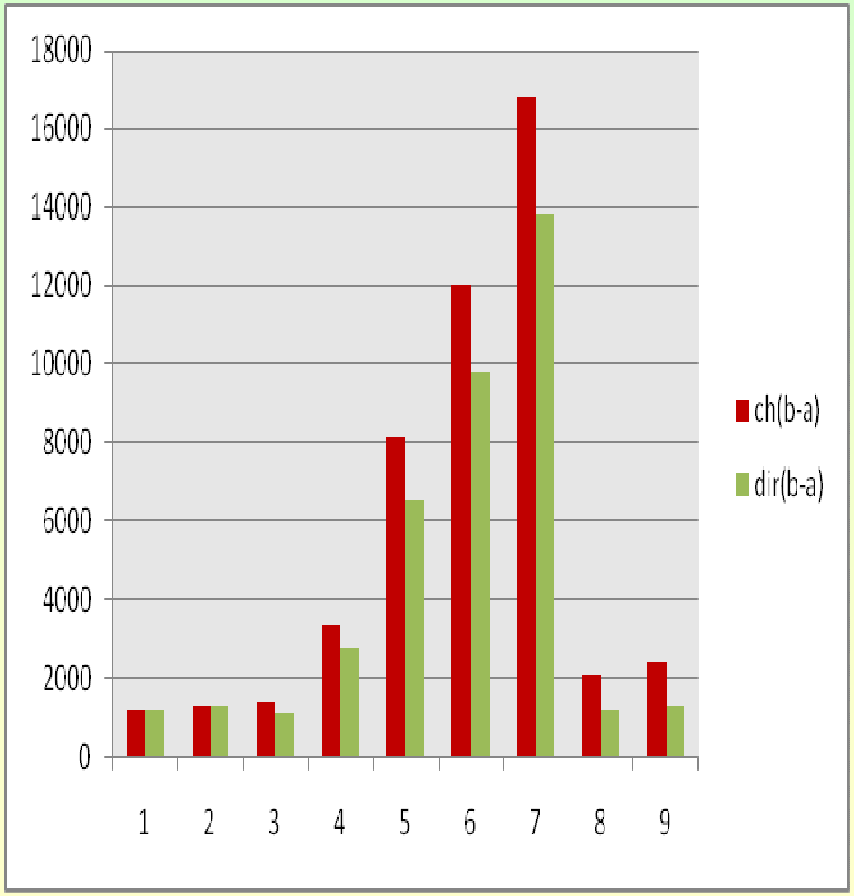
chvol = volume with chained deflators. $qp(0) = \Sigma q_t p_0$

component A



If B-A were a balancing item (e.g. net export), thus positive or negative


component B



6.4.3 (1) The sequence of volumes

see also **6.3 (8)**

direct Paasche	chain Paasche
$\sum q_1 p_0$	$\sum q_1 p_0$
$\sum q_2 p_0$	$\sum q_2 p_0 \frac{\sum p_0 q_1 \sum p_1 q_2}{\sum p_0 q_2 \sum p_1 q_1}$
$\sum q_3 p_0$	$\sum q_3 p_0 \frac{\sum p_0 q_1}{\sum p_0 q_3} \frac{\sum p_1 q_2}{\sum p_1 q_1} \frac{\sum p_2 q_3}{\sum p_2 q_2}$

general

$\sum q_t p_0$	$\sum q_t p_0 \frac{\sum p_0 q_1}{\sum p_0 q_t} \frac{\sum p_1 q_2}{\sum p_1 q_t} \cdots \frac{\sum p_{t-1} q_t}{\sum p_{t-1} q_{t-1}}$

6.4.3 (2) The sequence of volumes

direct Fisher	chain Fisher
$\sum q_1 p_0 \sqrt{\frac{\sum p_0 q_0}{\sum p_0 q_1} \frac{\sum p_1 q_1}{\sum p_1 q_0}}$	$\sqrt{\sum p_0 q_0 \sum p_1 q_1 \frac{\sum p_0 q_1}{\sum p_1 q_0}}$
$\sum q_2 p_0 \sqrt{\frac{\sum p_0 q_0}{\sum p_0 q_2} \frac{\sum p_2 q_2}{\sum p_2 q_0}}$	$\sqrt{\sum p_0 q_0 \sum p_2 q_2 \frac{\sum p_0 q_1 \sum p_1 q_2}{\sum p_1 q_0 \sum p_2 q_1}}$
$\sum q_3 p_0 \sqrt{\frac{\sum p_0 q_0}{\sum p_0 q_3} \frac{\sum p_3 q_3}{\sum p_3 q_0}}$	$\sqrt{\sum p_0 q_0 \sum p_3 q_3 \frac{\sum p_0 q_1 \sum p_1 q_2 \sum p_2 q_3}{\sum p_1 q_0 \sum p_2 q_1 \sum p_3 q_2}}$

general

$$\sum q_t p_0 \sqrt{\frac{\sum p_0 q_0}{\sum p_0 q_t} \frac{\sum p_t q_t}{\sum p_t q_0}}$$

$$\sqrt{\sum p_0 q_0 \sum p_t q_t \frac{\sum p_0 q_1 \sum p_1 q_2 \sum p_2 q_3}{\sum p_1 q_0 \sum p_2 q_1 \sum p_3 q_2} \cdots \frac{\sum p_{t-1} q_t}{\sum p_t q_{t-1}}}$$

$$= \sum q_t p_0 \sqrt{Q^{P}_{0t} / Q^{L}_{0t}}$$

6.5.1 (1) Chain index deflation and additivity: SNA and ESA on additivity

SNA (1993)

§16.56 Although desirable from an accounting viewpoint, additivity is actually a very restrictive property

$16.57 …publishing data only in the form of index numbers and not as values means abandoning any attempt to construct accounts at constant prices

$16.58 …there are effectively three ways of dealing with the ensuing non-additivity

- The first is simply to publish the non-additive constant price data as they stand without any adjustment….
- The second possibility it to distribute the discrepancies over the components at each level of aggregation …this procedure is not without its cost as the volume movements for the components are distorted. For certain types of analysis such distortions could be a serious disadvantage.
- A third possibility would be to eliminate the discrepancies by building up the values of the aggregates as the sum of the values … at each level of aggregation.

6.5.1 (2) Chain index deflation and additivity: SNA on additivity

SNA: different volumes (bad/additive, good/non-additive) for different kind of users:

$16.75 … it must be recognized that the lack of additive consistency can be a serious disadvantage for many types of analysis … It is therefore recommended that disaggregated constant price data should be compiled and published in addition to the chain indices for the main aggregates.

The need to publish two sets of data … should be readily appreciated by analysts …
Users whose interests are confined to a few global measures of real growth and inflation can be advised to utilize the chain indices and ignore the more detailed constant price estimates.

Given that a **new** method is usually introduced because it is a **better** method this means

- the less sophisticated users (those "who are confined…") get the better results (using the better [= chain index] method), whereas
- those who need better data (analysts, econometricians) get in addition figures gained with the old (traditional, inferior, abandoned) constant-prices-method.

6.5.1 (3) Chain index and additivity: ESA and BEA (USA) on additivity

The European position is quite similar. It is only that the Laspeyres-Paasche pair (P^P, Q^L) is still preferred to the Fisher index in deflation methodology:

ESA Council Regulation (No. 2223/96)

"... that disaggregated constant price data, i.e. direct valuation of current quantities at base-year prices, **are compiled in addition to the chain indices** for the main aggregates" (§ 10.66)

"...it will **have to be explained to users** why there is no additivity in the tables. The non additive 'constant price' data is published without any adjustment. This method is trans-parent and indicates to users the extent of the problem." (§ 10.67)

Bureau of Economic Analysis (BEA) US Department of Commerce

As usual when an index fails a "test" or axiom a debate breaks out coming to the point that passing the test is not desirable, or even noxious:

Ehemann, Katz & Moulton tried to play down the issue of additivity, contending
1. many types of analysis do not require additivity
2. traditional "fixed base" cross-sectional comparisons generally dubious
3. Deflation using Fisher indices is also approximately additive

6.5.1 (4) BEA (USA) on additivity (position of Ehemann, Katz & Moulton EKM)

BEA Non-additivity is not really a problem for chain index deflation

- additivity not required
  - often only the overall aggregate of interest
  - for many analyses only values (at current prices) relevant
- severe shortcomings of traditional methods outweigh and dwarf their advantage of additivity
  - the result of constant prices (traditional volumes) comparisons between aggregates say X_1 and X_2 is dubious and prone to error: $X_1 > X_2$ at prices of 2000, however $X_1 < X_2$ at prices of 2005
- an exact decomposition of percentage change (CPC)* of P^F deflation (like direct P^P deflation) is possible
  - use can be made of a formula of Dikhanov, mentioned also in Diewet who credits van Ijzeren for it (⇒)

note that the CPC of the i-th commodity does not only depend on $q_{i,.}$t and $q_{i,t-t}$ but also on two prices of i and the Fisher price index link P^F (so that weights are variable

$$\mathrm{CPC}_i^t = \frac{\left(p_{i,t-1} + \frac{p_{i,t}}{P_t^F}\right)\left(q_{i,t} - q_{i,t-1}\right)}{\left(\mathbf{p}_{t-1} + \mathbf{p}_t / P_t^F\right)' \mathbf{q}_{t-1}}$$

* of CPC_i contributions of components (the i th good) to the percent change in (the volume of) an aggregate

6.5.1 (5) Chain index and additivity: ESA and BEA (USA) on additivity

The basis of this formula can be found in Diewert's Lecture notes ch. 3, p. 17/18) reading as follows (in Diewert's notation):

"Consider the following N + 2 equations in the N + 2 unknowns, Q_F and P_F and pi*:
(i) $Q_F = \sum_{i=1}^{N} p_i^{*} q_i^{1} / \sum_{m=1}^{N} p_m^{*} q_m^{0}$;
(ii) $P_F Q_F = \sum_{i=1}^{N} p_i^{1} q_{i1} / \sum_{m=1}^{N} p_m^{0} q_m^{0}$;
(iii) $p_i^{*} = (1/2)p_i^{0} + (1/2)(p_i^{1}/P_F)$ for i = 1,…,N.
Show that the Q_F solution to the above equations is the Fisher ideal quantity index … . Thus (i) and (iii) show that the Fisher quantity index has an additive decomposition …, which is due to Van IJzeren (1987; 6). The ith reference price p_i^{*} is defined as
$p_i^{*} \equiv (1/2)p_i^{0} + (1/2)p_i^{1}/P_F(p_0,p_1,q_0,q_1)$ for i = 1,…,N and where P_F is the Fisher price index.

This decomposition was also independently derived by Dikhanov (1997). The Van IJzeren decomposition for the Fisher quantity index is currently being used by Bureau of Economic Analysis; see Moulton and Seskin (1999; 16) and Ehemann, Katz and Moulton (2002)."

The solution of this exercise and therefore the derivation of the CPC-formula is far from simple. By contrast the following presentation of Fisher's Q^F (or Q_F as above) in the textbook of Köves (p. 79) as an arithmetic (rather than geometric) mean is trivial and it also applies to P^F and the links of a (Fisher) chain index.

$$Q_{0t}^{F} = \frac{\left(Q_{0t}^{F} - Q_{0t}^{P}\right)Q_{0t}^{L} + \left(Q_{0t}^{L} - Q_{0t}^{F}\right)Q_{0t}^{P}}{Q_{0t}^{L} - Q_{0t}^{P}}$$

6.5.2 (1) Additive volumes and chain indices: suggestions of Claude Hillinger

Hillinger starts with a definitely wrong statement:

"**Only a uniform deflator applied to all prices** will produce expenditures satisfying the postulate ... (it) obviously **maintains the additivity** of nominal expenditures. The postulate is so elementary as to seem trivial, but no one appears to have thought of it."*

This rules out a deflation in volume terms and explains his preference for deflation in real income terms (a *trivial* solution to the additivity problem).

His central concept are AREVs (= aggregate real expenditure variation) defined in vector notation

$$\breve{y}^{t+1} - \breve{y}^{t} = \breve{\mathbf{p}}^{t+1}\mathbf{x}^{t+1} - \breve{\mathbf{p}}^{t}\mathbf{x}^{t}$$

where denotes a vector "reduced" to the base period value using the chained Marshal-Edgeworth (ME) price index

being the sum of

$$\tfrac{1}{2}\left(\breve{p}^{t+1} + \breve{p}^{t}\right)\left(x^{t+1} - x^{t}\right) = QV$$

QV = quantity variation and

$$\tfrac{1}{2}\left(x^{t+1} + x^{t}\right)\left(\breve{p}^{t+1} - \breve{p}^{t}\right) = PV$$

PV = Price variation

* his *postulate* is: real expenditures must reflect the exchange ratios in the market, i.e. current prices.

6.5.2 (2) Claude Hillinger's solution to additivity

Usage of $(\mathbf{x}^{t+1} + \mathbf{x}^{t})/2 = (\mathbf{q}_{t+1} + \mathbf{q}_{t})/2$ explains Hillinger's preference for the ME- Index

Definition of the "reduced" price vector $$\boxed{\breve{\mathbf{p}}^{t} = \frac{1}{\overline{P}_{0t}^{ME,C}}\mathbf{p}^{t}}$$

to make this clear it is worth-while spelling out P^{ME} in detail $$\overline{P}_{0t}^{ME,C} = \frac{\sum p_1(q_0+q_1)}{\sum p_0(q_0+q_1)}\frac{\sum p_2(q_1+q_2)}{\sum p_1(q_1+q_2)}\cdots\frac{\sum p_t(q_{t-1}+q_t)}{\sum p_{t-1}(q_{t-1}+q_t)}$$

Hillinger also discredited "additivity" by making a questionable comparison between volumes and quantities

> "...**quantities** are generally not additive ... it makes no sense to add heterogeneous units." (**Volumes**) "remain quantity measures, and **adding them up is**, if anything, **misleading** ... there is no reason to seek additivity for quantities and its absence cannot therefore be a source of concern" (p. 16)

If this were correct, also values (at *current* prices) would be meaningless: they also represent quantities only related to different prices than volumes. And if his *postulate* (real expenditures must reflect current prices) were correct why should we deflate at all?

Moreover: "A serious difficulty with this system is that the level of subaggregate k need not be positive, even if all of its components and their prices are positive" (Ehemann/Katz/Moulton)

6.5.3 (1) Additivity of volumes and chain index deflation: Balk's solution

An attempt to reconcile chain-index-deflation and additivity

BR = Bert M. Balk, Utz-Peter Reich, Additivity of National Accounts Reconsidered
Journal of Economic and Social Measurement 33 (2008) pp 165 – 178 (version June 2007)

BR acknowledge that there is a fundamental inconsistency

"a mathematical impossibility result; between two conflicting goals"

however

"realism of the price system … has been accorded over additivity of values yielding coherent national accounts

they recommend, however, the compromise

volume oriented **approach** for period t (K indices for the K sub-aggregates)

and

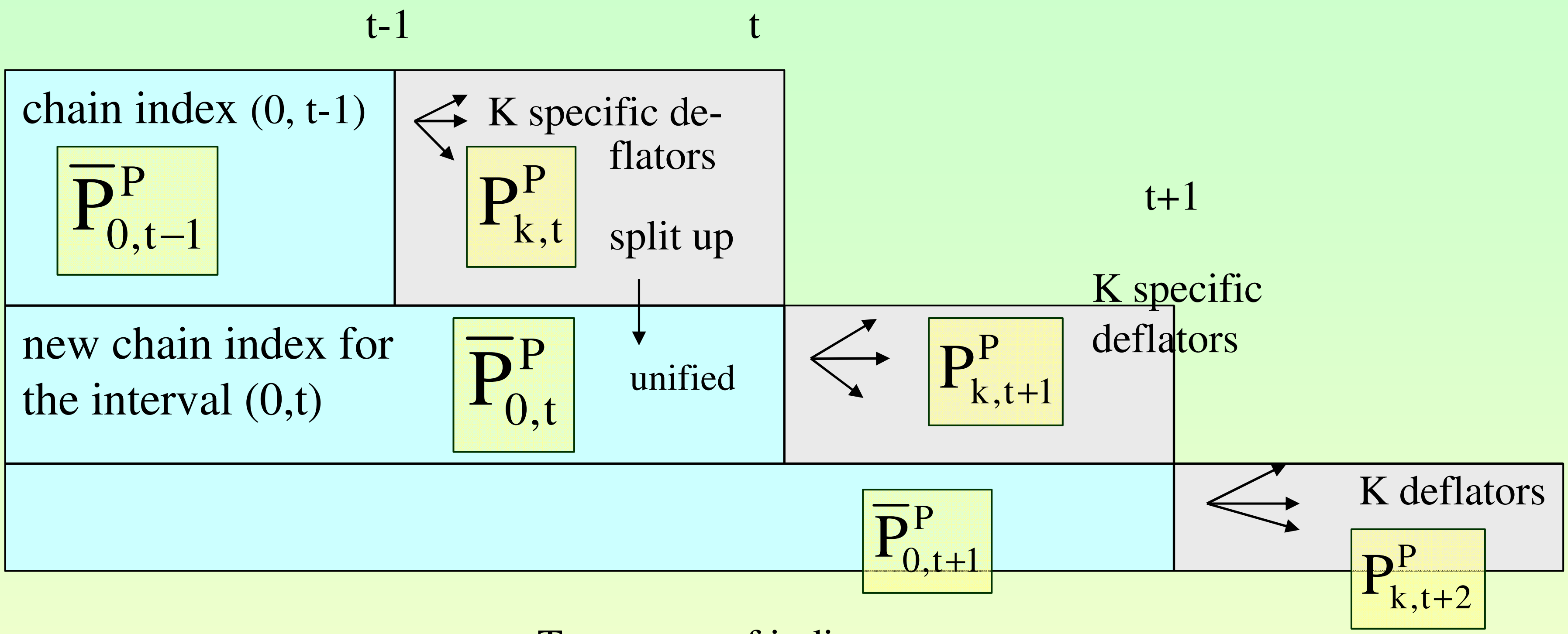
real income approach for all t-1 periods prior to t (one single chain index for the total aggregate over the interval [0, t-1])

$P^{P}_{k,t}$ specific for sub-aggregate k

$\overline{P}^{P}_{0,t-1}$ for the total aggregate

$$P^{*}_{k,0t} = P^{P}_{k,t}\overline{P}^{P}_{0,t-1}$$

6.5.3 (2) Additivity and chaining: Balk's solution



t-1

t

chain index (0, t-1) $\overline{P}^{P}_{0,t-1}$

K specific de-flators $P^{P}_{k,t}$

split up

new chain index for the interval (0,t) $\overline{P}^{P}_{0,t}$

unified

t+1

K specific deflators $P^{P}_{k,t+1}$

$\overline{P}^{P}_{0,t+1}$

K deflators $P^{P}_{k,t+2}$

Two types of indices

chain index for the total aggregate $\overline{P}^{P}_{0,t-1}$

chained but not used for deflation

sub-aggregate specific $P^{P}_{k,t}$ → for deflation, will not be chained

$\overline{P}^{P}_{k,0,t-1}$

does not exist

$$\boxed{P^{*}_{k,0t} = P^{P}_{k,t}\overline{P}^{P}_{0,t-1}}$$

6.5.3 (3) Additivity and chain index deflation: solution of Balk and Reich

$V_{k,t}$ value ($\Sigma p_t q_t$) of the (sub-) aggregate k and $V_t = \Sigma V_{k,t}$ k = 1, 2, …, K

	subaggregate k	aggregate (alle K)
links of the chain index	$P^P_{k,t} = \frac{\sum p_{kjt}q_{kjt}}{\sum p_{kj,t-1}q_{kjt}}$	$P^P_t = \frac{\sum\sum p_{kjt}q_{kjt}}{\sum\sum p_{kj,t-1}q_{kjt}} = \left(\sum \frac{1}{P^P_{k,t}}\frac{V_{k,t}}{V_t}\right)^{-1}$
usage	these indices are **not** used for **chaining** $\overline{P}^P_{k,0t} = \prod_{\tau=1}^{t} P^P_{k,\tau}$,	these indices are **chained** (general price level as chain index) **but** they are not used for **deflation** $\overline{P}^P_{0,t} = P^P_t \overline{P}^P_{0,t-1}$
The deflator	$P^*_{k,0t} = P^P_{k,t}\overline{P}^P_{0,t-1}$	

does not exist

$$\overline{P}^P_{0,t-1} = \prod_{\tau=1}^{t-1} P^P_\tau$$

specific for sub-aggregate k

for all sub-aggregates

Chain index and deflator no longer identical

6.5.3 (4) Interpretation of the BR volumes

The sequence of (BR-) volumes Q^{BR} for the k-th sub-aggregates more complicated: the ratio Q_t/Q_{t-1} (and difference $Q_t - Q_{t-1}$) "*is a less meaningful measurement tool*" However Qt/Qt-1 – 1 is the growth rate of real GDP

With **direct** Paasche deflators and Laspeyres-volume indices Q^L we have (j = 1, …, n_k)

Difference	$Q^L_{kt} - Q^L_{k,t-1} = \sum_j p_{kj0}\left(q_{kjt} - q_{kj,t-1}\right)$
Ratio	$Q^L_{k,t} \big/ Q^L_{k,t-1} = \sum_j p_{kj0} q_{kj,t} \big/ \sum_j p_{kj0} q_{kj,t-1}$ ← mean of q_t/q_{t-1} terms

However, in the Balk/Reich (BR) methodology

and different prices

Difference	$Q^{BR}_{kt} - Q^{BR}_{k,t-1} = \sum_j p_{kj,t-1} q_{kj,t} \big/ \overline{P}^P_{0t} - \sum_j p_{kj,t-2} q_{kj,t-1} \big/ \overline{P}^P_{0,t-1}$
Ratio	$\dfrac{Q^{BR}_{kt}}{Q^{BR}_{k,t-1}} = \dfrac{\sum_j p_{kj,t-1} q_{kj,t}}{P^P_t \sum_j p_{kj,t-2} q_{kj,t-1}} = \left(\dfrac{P^P_{k,t}}{P^P_t}\right) \dfrac{\sum_j p_{kj,t-1} q_{kj,t}}{\sum_j p_{kj,t-1} q_{kj,t-1}}$

path dependence (history of prices and quantities until t-1)

this second term is a mean of q_t/q_{t-1} terms; in numerator and denominator the same prices (of t-1)

6.5.3 (5) Values (V) and volumes (Q), Why additivity?

BR volumes Definition and additivity

individual volumes	$Q^{BR}_{k,t} = \frac{V_{k,t}}{P^{*}_{k,0t}} = \frac{V_{k,t}}{P^{P}_{k,t}} \frac{1}{\overline{P}^{P}_{0,t-1}} = \frac{V_{k,t}}{P^{P}_{k,t}} C^{-1}$
additivity	$\sum_k Q^{BR}_{k,t} = \sum_k \frac{V_{kt}}{P^{*}_{k,0t}} = C^{-1} V_t \sum_k \left(\frac{1}{P^{P}_{k,t}} \frac{V_{kt}}{V_t} \right) = \left(V_t C^{-1}\right)\left(P^{P}_t\right)^{-1} = \frac{V_t}{\overline{P}^{P}_{0t}}$

Sequence of volumes of sub-aggregate k: $Q_{k,2}$, $Q_{k,3}$, $Q_{k,4}$ …

direct Paasche = Laspeyres quantity	$\sum_j p_{kj0} q_{kj2}$	$\sum_j p_{kj0} q_{kj3}$	$\sum_j p_{kj0} q_{kj4}$
BR-volumes	$\sum_j p_{kj1} q_{kj2} \big/ \overline{P}^{P}_{01}$	$\sum_j p_{kj2} q_{kj3} \big/ \overline{P}^{P}_{02}$	$\sum_j p_{kj3} q_{kj4} \big/ \overline{P}^{P}_{03}$

Sequence of BR-volumes appears less meaningful

6.5.4 (1) Critique and example: BR-volumes violate proportionality

The example shows that BR deflation violates proportionality (thus also identity)

commodities 1 and 2 belong to aggregate A, 3 and 4 to aggregate B

Direct deflation (Laspeyres volume) Paasche meets proportionality in the q_t

i	p_{i0}	q_{i0}	p_{i1}	q_{i1}	p_{i2}	q_{i2}
1	5	10	8	8	5	10
2	8	6	10	7	8	6
3	3	4	5	6	3	8
4	6	8	8	9	6	16

values $\Sigma q_t p_t$

	0	1	2
A	98	134	**98**
B	60	102	**120**
Σ	158	236	**218**

direct Paasche deflator

	(P_{01}) t = 1	(P_{02}) t = 2
A	134/96 = 1,396	1
B	102/72 = 1,417	1
Σ	236/168 = 1,405	1

▨ this part is needed only for the chain index

volumes $\Sigma q_t p_0$

	t = 1	t = 2
A	96	**98**
B	72	**120**
Σ	168	**218**

6.5.4 (2) BR-volumes violate proportionality

Deflation according to BR (Balk/Reich)

grey what did not change

Paasche chain indices

	(P_{01}) t=0 → t = 1	(P_{12}) t = 1 → t = 2
A	134/96 = 1,396	98/140 = 0,7000
B	102/72 = 1,417	120/168 = 0,7143
Σ	236/168 = 1,405	218/308 = 0,7078

Deflator in period 2

A	0,7000*1,405 = 0, 9833
B	0,7143*1,405 = 1,0034
Σ	0,7078*1,405 = 0,9943

$$P^*_{k,t} = P^P_{k,t}\overline{P}^P_{0,t-1}$$

$$\overline{P}^P_{0t}$$

values (as before)

	0	1	2
A	98	134	**98**
B	60	102	**120**
Σ	158	236	**218**

volume

	t = 1	t = 2
A	96	**99,661**
B	72	**119,593**
Σ	168	**219,254**

no identity, although additivity is given

6.5.4 (3) Variant of the numerical example: high prices in the intermediate period

some new prices p_{i1} ceteris paribus in blue

grey as before

i	p_{i0}	q_{i0}	p_{i1}	q_{i1}	p_{i2}	q_{i2}
1 2	5 8	10 6	18 20	8 7	5 8	10 6
3 4	3 6	4 8	15 8	6 9	3 6	8 16

values

	0	1	2
A	**98**	284	**98**
B	**60**	168	**120**
Σ	158	452	**218**

BR-deflator index in t=2

A	(98/300)*(452/168) = 0, 8788
B	(120/248)*(452/168) = 1,3018
Σ	(218/548)*(452/168) = 1,0703

volumes

	t = 1	t = 2
A	96	**111,504**
B	72	**92,177**
Σ	168	**203,681**

P_{01} = 452/168 = 2,6905 enormous rise of prices (0 to 1), then prices were declining (by 32 or 48% respectively) to their original level

no identity

6.5.4 (4) Solution of Balk and Reich: summary

1. Chain index and deflator no longer identical

   aggregate-specific indices are not chained, and
   chained indices are not used for deflation

2. Differences/ratios (growth rates) of successive volumes (year-on- year) difficult to interpret

   they do not represent a pure volume change; different prices in numerator and denominator, that is no volumes at constant prices and path-dependent, BR: "less meaningful"

3. Volumes are not proportional in the quantities q_t

   Same prices and same (or proportional) quantities → yet different volumes; the violation of proportionality is particularly pronounced when prices are exceptionally high/low in the intermediate period(s).

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