Generalized Statistical Means and New Price Index Formulas, Notes on some unexplored index formulas, their interpretations and generalizations

Peter von der Lippe

University of Duisburg-Essen, Germany

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The theory of (increasingly more generalized types of) statistical means can be used to create a plethora of index formulas. Some of them are new and some were indeed discussed in the past but fallen into oblivion, because their rationale was not well understood. Surprisingly many possess interesting interpretations and attractive properties that deserve being unveiled. We begin with unweighted indices with implications to what now is called "low level aggregation" and proceed to weighted index formulas that lend themselves to productive generalizations and thereby to some new formulas. It turns out that contrary to popular belief the Laspeyres and Paasche formula are not equally well justified and that some indices from the more comprehensive system of statistical means are attractive regarding their economic interpretation and how they are related to indices of "quantity" and purchasing power of money.

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* Average of ratios formulas ** Ratios of averages

1. Unweighted indices (low level aggregation)

1.1. Unweighted indices and implicit quantities

Carli's index $P_{0t}^{C} = \frac{1}{n}\sum \frac{p_{it}}{p_{i0}}$ was widely used in early price level measurement in the 19th century.[1] It was considered desirable to account for the different relative "importance" of the n goods in this formula. However, difficulties with aggregating over widely different types of goods and their quantities[2] (we cannot add kilograms of vegetables, liters of fuel, yards of cloth, hours of services etc) have led some authors to adopt the idea of *reciprocal prices as "implicit quantities"*: obviously $1/p_{i0}$ is the amount of good i you get for one currency unit (say 1€, or 1$) at the base period.[3] With such "implicit quantities" a mean of price relatives becomes a ratio of expenditures. When exactly 1€ was spent for each good in the base period, we have a basket of n goods and an expenditure (or cost for a basket) of E_0 = n€ in the base period and Carli's index P^C compares $E_t = \sum_{i=1}^{n} p_{it}(p_{i0})^{-1}$ with $E_0 = \sum p_{i0}(p_{i0})^{-1} = n$, because

[1] Laspeyres and some other authors made extensively use of this formula, then also known as "Sauerbeck index" in these days, and of Sauerbeck's price statistics for the British foreign trade. It was only in the 20th century owing to Walsh 1901 that it became generally known that the formula actually was much older and originated from Gian Rinaldo Carli (1720 -1795).

[2] A problem inherent in unit-value-indices (see below part 4)

[3] Note reciprocal prices are not quantities. Just as a price is measured in currency units per unit of the quantity in question (say € per gram) so $1/p_{i0}$ is a gram per € expression.

$$(1.1)\quad P_{0t}^{C}=\frac{E_t}{E_0}=\frac{\sum p_{it}\left(\frac{1}{p_{i0}}\right)}{n}=\frac{\frac{1}{n}\sum p_{it}\left(\frac{1}{p_{i0}}\right)}{\frac{1}{n}n}=\frac{1}{n}\sum\frac{p_{it}}{p_{i0}}.$$

It may also be interpreted as *comparing weighted average prices* (with reciprocal prices or implicit quantities as weights), $\bar{p}_t^C = E_t/n$ and $\bar{p}_0^C = E_0/n = 1$. With this (admittedly somewhat farfetched) notion of "implicit quantities" an average of price ratios (or **AOR-formula**) can easily be translated into a ratio of average expenditures/prices or **ROA formula**. Or in other words: an unweighted index may be viewed as implicitly, i.e. with reciprocal prices weighted. The same logic, if applied to an unweighted *harmonic* mean will lead to

$$(1.2)\quad P_{0t}^{H}=\left(\frac{1}{n}\sum\left(\frac{p_{it}}{p_{i0}}\right)^{-1}\right)^{-1}=\frac{n}{\sum\frac{p_{i0}}{p_{it}}}=\frac{\sum p_{it}(p_{it})^{-1}}{\sum p_{i0}(p_{it})^{-1}}=\frac{E_t^*}{E_0^*}$$ where E_t^* and E_0^* are expenditures

with current period "implicit quantities" $1/p_{it}$. That $P_{0t}^H = \left(P_{t0}^C\right)^{-1}$ means that P^H is the "time antithesis" (as Fisher would have put it) of P^C (and vice versa). Note that using *relative* prices or "price shares" (or price quotas p/Σp) rather than *reciprocal* prices we also can transform a ROA type index such as Dutot's index P^D into a weighted AOR index like of the P^C or P^H type

$$P_{0t}^{D}=\frac{\sum p_{it}}{\sum p_{i0}}=\frac{\frac{1}{n}\sum p_{it}}{\frac{1}{n}\sum p_{i0}}=\frac{\bar{p}_t}{\bar{p}_0}$$ as weighted (P^C-type) AOR index $P_{0t}^{D}=\sum\frac{p_{it}}{p_{i0}}\frac{p_{i0}}{\sum p_{i0}}$ and

$$P_{0t}^{D}=\left(\frac{\sum p_{i0}}{\sum p_{it}}\right)^{-1}=\frac{\bar{p}_t}{\bar{p}_0}\rightarrow P_{0t}^{D}=\left(\sum\frac{p_{i0}}{p_{it}}\frac{p_{it}}{\sum p_{it}}\right)^{-1}$$ as weighted P^H type AOR index.[4]

ROA form		**weights**		**AOR form**
ratio of average expenditures/prices* as for example P^D	→	*relative* prices p/Σp	→	average of price ratios [= price relatives] as for example P^C and P^H
	←	*reciprocal* prices (as implicit quantities)	←	

* equal implicit quantities 1/n for each good

Obviously in (1.1) and (1.2) the "quantity" of each good is indirectly proportional to its price, so expensive goods are less, and cheaper goods are more represented in such an index. This suggests taking some average of such implicit quantity weights (leading to Young's index P^Y).

The formulas P^C (Carli) and P^H (harmonic) were mainly discussed (and rejected) because of their failing the time reversal test since $P_{t0}^C = \left(P_{0t}^H\right)^{-1} \geq \left(P_{0t}^C\right)^{-1}$ and $P_{t0}^H = \left(P_{0t}^C\right)^{-1} \leq \left(P_{0t}^H\right)^{-1}$.

1.2. Crossing of formulas and weights

A quite natural idea is to take a mean of P^C and P^H. The *geometric* mean of P^C and P^H, that is

$$(1.3)\quad P_{0t}^{CSWD}=\sqrt{P_{0t}^{C}P_{0t}^{H}}=\sqrt{\frac{1}{n}\sum\frac{p_{it}}{p_{i0}}:\frac{n}{\sum\frac{p_{i0}}{p_{it}}}}=\sqrt{\frac{\sum p_{it}/p_{i0}}{\sum p_{i0}/p_{it}}}$$, is known as CSWD-index[5] and

for approximating the time reversible index of Jevons $P_{0t}^{J}=\sqrt[n]{\prod\frac{p_{it}}{p_{i0}}}$. It can easily be seen that

[4] So taking a harmonic mean of the price relatives with weights $p_{it}/\Sigma p_{it}$ will again yield P^D. Unlike Carli's arithmetic mean index the harmonic mean P^H of price relatives never gained much attention.

P^J and P^{CSWD} pass the time reversal test, since $1/P^J_{t0} = P^J_{0t}$ $P^{CSWD}_{t0} = 1/P^{CSWD}_{0t}$.[6] In a similar manner we can also take averages of weights $P^{CW}_{0t} = \frac{\sum p_{it} W_i}{\sum p_{i0} W_i}$ where the part of weights W_i can be taken by

- $A_i = \frac{1}{2}\left(\frac{1}{p_{i0}} + \frac{1}{p_{it}}\right)$ (arithmetic) giving $P^{CW(A)}_{0t} = \frac{\sum p_{it} A_i}{\sum p_{i0} A_i} = \frac{1 + P^C_{0t}}{1 + P^C_{t0}}$
- $H_i = \frac{2}{p_{i0} + p_{it}}$ (harmonic), yielding $P^{CW(H)}_{0t} = \frac{\sum p_{it} H_i}{\sum p_{i0} H_i} = \frac{\Sigma(p_{it}/(p_{i0}+p_{it}))}{\Sigma(p_{i0}/(p_{i0}+p_{it}))}$, and
- $G_i = \sqrt{(p_{i0} p_{it})^{-1}} = \sqrt{A_i H_i}$ (geometric) $\rightarrow$ $P^{CW(G)}_{0t} = \frac{\sum p_{it} G_i}{\sum p_{i0} G_i} = \frac{\sum \sqrt{p_{it}/p_{i0}}}{\sum \sqrt{p_{i0}/p_{it}}} = P^Y_{0t}$.

In all three cases both, numerator $\Sigma p_{it} W_i$ and denominator $\Sigma p_{i0} W_i$ may be viewed as expenditures, or (upon division by n) as weighted average prices. So crossed implicit weight (CW) indices can be given an ROA-interpretation in addition to an AOR interpretation as for example $P^{CW(H)}_{0t} = \frac{\sum \frac{p_{it}}{p_{i0}} (p_{i0}/(p_{i0}+p_{it}))}{\sum (p_{i0}/(p_{i0}+p_{it}))}$ is an average of price relatives p_{it}/p_{i0}. The most interesting index is $P^{CW(G)}$ as it coincides with an index P^Y suggested by Allyn A. Young (1923)[7].

1.3. Index of Allyn A. Young P^Y

Young proposed the following seemingly weird and unmotivated formula

$$(1.4) \quad P^Y_{0t} = \frac{\sum p_t \sqrt{\frac{1}{p_0 p_t}}}{\sum p_0 \sqrt{\frac{1}{p_0 p_t}}} = \frac{\sum \sqrt{\frac{p_t}{p_0}}}{\sum \sqrt{\frac{p_0}{p_t}}}$$

For Young it was the "best unweighted index"; it meets both, time reversibility and linear homogeneity:

linear homogeneity	time reversal test	
	yes	no
yes	Y	C, H, CSWD
no	CW(A), CW(H)	

Given this favourable situation of P^Y the question may arise why this index did not find attention and why it obviously is completely fallen into oblivion. We think that this is due to the fact that its rationale is not well understood when it was proposed in 1923.[8] Nobody seems to know by which underlying "logic" (of crossing inverse prices as implicit quantities) we would arrive at Young's index P^Y. So some remarks to its rationale should be pertinent here:Young found that "base year weighting" in $P^C_{0t} = \sum p_t (p_0)^{-1} / \sum p_0 (p_0)^{-1}$ (with weights in the way of inverse base period prices or implicit quantities) tends to "overweight rising prices", while

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[5] As acronym for Carruthers, Selwood, Ward and Dalen.
[6] P^J has both, an AOR and a ROA interpretation (taking *geometric* means of prices averages in the ROA).
[7] This Young is not to confound with Arthur Young 1812, also often quoted in index theory literature. Unlike $P^{CW(G)}$ the formulas $P^{CW(A)}$ and $P^{CW(H)}$ possibly rightly did not receive much attention.
[8] Irving Fisher 1927; 530f later called P^Y "an ingenious anomaly, scarcely classifiable" (in the scheme of Fisher's book) and "a scientific curiosity".

$P_{0t}^{H} = \sum p_t (p_t)^{-1} / \sum p_0 (p_t)^{-1}$, tends to underweight them. Thus he was quite naturally lead to seek a compromise using the geometric mean $\sqrt{\frac{1}{p_0 p_t}}$. Later Bert Balk rediscovered Young's formula and called it Balk-Walsh (BW) index, because with *explicit* (quantity) weights we get Walsh's formula $P_{0t}^{W} = \frac{\sum p_t \sqrt{q_0 q_t}}{\sum p_0 \sqrt{q_0 q_t}}$ which bears some resemblance to Young's formula P^Y based on *implicit* (inverse prices) rather than explicit weights. Another rediscovery of P^Y took place when Jens Mehrhoff – in a short note he contributed to von der Lippe 2007; 45f – looking for a linear index able to approximate P^{CSWD} and thereby P^J. He called it "hybrid" and later "BMW (Balk-Mehrhoff-Walsh) index", not knowing that it coincides with P^Y and he also remarked (like von Bortkiewicz and even already Young before), that P^Y not only has a ROA interpretation (indicated in (1.4) with weighted means of prices) but also an AOR interpretation as follows

(1.5) $$P_{0t}^{Y} = \frac{\sum \frac{p_t}{p_0} \sqrt{\frac{p_0}{p_t}}}{\sum \sqrt{\frac{p_0}{p_t}}},$$

with $\sqrt{p_0 / p_t}$ as somewhat awkward weights. As mentioned already Young well appreciated this property of possessing a double interpretation (AOR in 1.5 and ROA in 1.4) of his index formula when he said:

> "In a way Professor Fisher is right in holding that all true index numbers are averages of ratios. But I should prefer to say that all true index numbers are at once averages of ratios and ratios of aggregates." (Young 1923; 359).

Young also saw that his index meets the time reversal test but not the circular test, because

$$P_{02}^{Y} = \frac{\sum \sqrt{\frac{p_2}{p_0}}}{\sum \sqrt{\frac{p_0}{p_2}}} \neq \frac{\sum \sqrt{\frac{p_1}{p_0}}}{\sum \sqrt{\frac{p_0}{p_1}}} \frac{\sum \sqrt{\frac{p_2}{p_1}}}{\sum \sqrt{\frac{p_1}{p_2}}}$$

, and finally (interesting in view of Mehrhoff's paper) Young also noted about P^Y "In general it will agree very closely with the geometric average" (357).

2. Weighted indices: Average of ratios formulas (AOR), part I:

2.1. Fisher's system and Vogt's generalized antiharmonic mean

A distinction can be made between first, second, and third order index formulas (Köves 1983; 28ff, v. d. Lippe 2007; 54). "First" means unweighted indices, "second" relates to Irving Fisher's system of AOR-formulas and "third" to "crossing" of second generation indices (for example Fisher's ideal index as result of crossing Laspeyres and Paasche). Fisher introduced

- *six **types of*** (unweighted) ***means*** (of price relatives), of which only arithmetic, harmonic and geometric means are worth being considered,[9] and
- *four* ***methods*** **of** ***weighting*** (I through IV, of which he called two, II and III "hybrid")

by which he arrived at the following system:[10]

[9] The three remaining means are median, mode, and the "aggregative" index as Fisher called it (like the Dutot index $\Sigma p_t / \Sigma p_0$). They are no longer of any interest.

[10] Note that some of the combinations are identical due to inherent relations between the arithmetic and the harmonic mean: both index formulas, P^L and P^P can be expressed in two ways, using pure and hybrid weights. It will soon be shown that all the remaining indices can be written in two different ways as well (see tab. 3).

Table 1

means	methods of weighting			
	I: base p_0q_0	II: hybrid p_0q_t	III: hybrid p_tq_0	IV: current p_tq_t
arithmetic	P_{0t}^{L} Laspeyres	P_{0t}^{P} Paasche	P_{0t}^{AH} see eq. 2.1	P_{0t}^{PA} Palgrave
harmonic	P_{0t}^{HB} see eq. 2.12	P_{0t}^{HH} see (2.11)	P_{0t}^{L} Laspeyres	P_{0t}^{P} Paasche

The highlighting with green n orange color is meant as hint to related fields (in the same color) of table 2

The combination gives six different "second generation" index-formulas P^L, P^P, P^{AH}, P^{PA}, P^{HB}, and P^{HH} (of which only the indices of Laspeyres and Paasche are well known, P^{PA} refers to the index of Palgrave, and the other labels are our own: HB = harmonic base, HH = harmonic hybrid, AH = arithmetic hybrid). The indices P^{HB} and P^{PA} are a bit more interesting than the indices P^{AH} and P^{HH} (which are rarely if ever seriously considered), so P^{HB} and P^{PA} will be discussed in more detail below (section 3).[11]

Obviously P_{0t}^{AH} is related to the so called quadratic mean (QM) as follows

$$(2.1)\quad P_{0t}^{AH}=\sum\frac{p_t}{p_0}\frac{p_tq_0}{\sum p_tq_0}=\frac{\sum p_t^2q_0/p_0}{\sum p_tq_0}=\frac{\sum\left(p_t^2/p_0^2\right)p_0q_0/\sum p_0p_0}{\sum p_tq_0/\sum p_0q_0}=\frac{(P_{0t}^{QM})^2}{P_{0t}^{L}}$$

where P_{0t}^{QM} is the quadratic and P_{0t}^{L} the arithmetic mean of price relatives with type I weights.

The system of tab. 1 can be generalized in view of

- the fact that all means, harmonic ($\bar{x}_H$) geometric ($\bar{x}_G$) arithmetic ($\bar{x}$) and quadratic ($\bar{x}_{QM}$) are special cases of $\bar{x}_p(r)$, the ***power mean*** (*moment mean* or generalized mean) of degree r given by

  $$(2.2)\quad \bar{x}_P(r)=\left[w_1x_1^r+w_2x_2^r+\ldots+w_mx_m^r\right]^{1/r}$$,[12] and

- the (weighted) ***antiharmonic mean*** (denoted by $\overline{H}$ to avoid confusion with AH denoting arithmetic hybrid) of x-values with weights w discussed in Vogt 1979

  $$(2.3)\quad \bar{x}_{\overline{H}}=\frac{\sum x_i^2w_i}{\sum x_iw_i}=\left(\bar{x}_Q\right)^2/\bar{x}$$

  that is the ratio of the squared quadratic and the arithmetic mean,[13] so that P^{AH} is in fact an antiharmonic mean using weights of type I. This motivates an extension of the scheme above as follows:

Table 1a

mean	methods of weighting			
	I: base p_0q_0	II: hybrid p_0q_t	III: hybrid p_tq_0	IV: current p_tq_t
antiharmonic	$P_{0t}^{AH}=P_{0t}^{GAH}(0,1)$	$P_{0t}^{\overline{H}2}=P_{0t}^{PA}$	$P_{0t}^{\overline{H}3}\rightarrow(2.10)$	$P_{0t}^{\overline{H}4}=P_{0t}^{GAH}(t,1)$

[11] In Germany Neubauer 1998 brought them into play and v. d. Lippe 2000 discussed their properties.

[12] The special cases are r = -1 harmonic mean, r → 0 geometric mean, r = 1 the arithmetic mean and r = 2 quadratic mean. According to a corollary, first proven by Cramer the power mean is a monotonous function in the parameter r such that harmonic ≤ geometric ≤ arithmetic ≤ quadratic (equality applies when all x-values are identical). From this follows $P^{HH} < P^P < P^{PA}$ and $P^{HB} < P^L < P^{AH}$ in tab. 1

[13] So P^{AH} is not only arithmetic hybrid but also antiharmonic (with the non-hybrid or "pure" weight system I).

- the even more general concept of the ***generalized antiharmonic or "GAH" mean*** (again brought into play by Vogt)

$$(2.4)\qquad P_{0t}^{GAH}(j,k)=\frac{\sum s_j\left(\frac{p_t}{p_0}\right)^{k+1}}{\sum s_j\left(\frac{p_t}{p_0}\right)^{k}}$$

with expenditure shares $s_{ij} = p_{ij}q_{ij}/\Sigma p_{ij}q_{ij}$ (j = 0, t, hybrid "weights" will henceforth no longer be considered; summation takes place over i) taking the place of the weights w_i above.
k = 1 is the "usual" antiharmonic mean introduced above.

and the GAH mean allows some interesting extensions of table 1.

Table 2

	k = -2	k = -1 harmonic	k = -1/2	k = 0 arithmetic	k = + 1/2	k = + 1 antiharmonic
j = 0 weights I	$P_{0t}^{GAH}(0,-2)$	P_{0t}^{HB}	$P_{0t}^{GAH}(0,-\frac{1}{2})$	P_{0t}^{L}	$P_{0t}^{GAH}(0,\frac{1}{2})$	P_{0t}^{AH}
j = t weights IV	P_{0t}^{HH}	P_{0t}^{P}	$P_{0t}^{GAH}(t,-\frac{1}{2})$	P_{0t}^{PA}	$P_{0t}^{GAH}(t,\frac{1}{2})$	$P_{0t}^{GAH}(t,1)$

In table 2 there are no longer any hybrid weights Together with two additional rows for hybrid weights (type II and III) we would get with k = -1 (orange) and k = 0 (green) the complete table 1 as a special case (subset) of table 2.

Before going into details of table 1a and 2 it may be useful to introduce certain fictitious quantities in the following table

Table 1b

	deflated	inflated
q_0	$q_0^D=\frac{q_0}{p_t/p_0}=\frac{q_0p_0}{p_t}$	$q_0^I=\frac{q_0p_t}{p_0}$
q_t	$q_t^D=\frac{q_t}{p_t/p_0}=\frac{q_tp_0}{p_t}$	$q_t^I=\frac{q_tp_t}{p_0}$

The red double arrow means q_t^I is the "time antithesis" (I. Fisher) of q_0^D (and vice versa), and so is q_t^D of q_0^I. We will come back to this table 1b in section 3 in our attempt to give a meaningful interpretation to P_{0t}^{HB} and P_{0t}^{PA}.

The missing formula $P_{0t}^{\bar{H}2}$ in tab. 1a now is

$$(2.9)\qquad P_{0t}^{\bar{H}2}=\frac{\sum\frac{p_t^2}{p_0^2}\frac{p_0q_t}{\sum p_0q_t}}{\sum\frac{p_t}{p_0}\frac{p_0q_t}{\sum p_0q_t}}=\frac{\sum p_tq_t^I}{\sum p_tq_t}=P_{0t}^{PA}$$ and for $P_{0t}^{\bar{H}3}$ we get

$$(2.10)\qquad P_{0t}^{\bar{H}3}=\frac{\sum\frac{p_t}{p_0}(p_tq_0^I)}{\sum(p_tq_0^I)}$$

which does not seem to be a useful formula. This also applies to P^{HH} which is given by

$$(2.11)\qquad P_{0t}^{HH}=\frac{\sum p_0q_t}{\sum p_0q_t^D}$$, or written as a GAH (t, -2) index $$P_{0t}^{HH}=\frac{\sum\left(\frac{p_t}{p_0}\right)^{-1}p_tq_t}{\sum\left(\frac{p_t}{p_0}\right)^{-2}p_tq_t}$$

which bears some resemblance with

(2.12) $P_{0t}^{HB} = \frac{\sum p_0 q_0}{\sum p_0 q_0^D}$, or written as a GAH (0, -1) index $P_{0t}^{HB} = \frac{\sum \left(\frac{p_t}{p_0}\right)^0 p_0 q_0}{\sum \left(\frac{p_t}{p_0}\right)^{-1} p_0 q_0}$.

From tab. 2 two remaining formulas may be of some interest:

$P_{0t}^{GAH}(0,-2)$	$\frac{\sum p_t \left(p_0 q_0^D / p_t\right)}{\sum p_0 \left(p_0 q_0^D / p_t\right)}$	harmonic mean weights $p_0 q_0^D$ (deflated base period)
$P_{0t}^{GAH}(t,1)$	$\frac{\sum p_t \left(p_t q_t^I / p_0\right)}{\sum p_{0t} \left(p_t q_t^I / p_0\right)}$	arithmetic mean weights $p_t q_t^I$ (inflated current period)

In order to compare this with tab. 1 note that $P_{0t}^{AH} = P_{0t}^{GAH}(0,1)$ and $P_{0t}^{HH} = P_{0t}^{GAH}(t,-2)$

Surprisingly: while the indices P^{AH} and P^{HH} were introduced already above as indices with *hybrid* weights, it turns out now in tab. 2 that they also emerge as indices with "pure" weights I and IV. So types of means and types of *weights* are somehow related. As to the still remaining four indices of tab. 2:

	k = -1/2	k = +1/2
j = 0	$P_{0t}^{GAH}(0,-\tfrac{1}{2}) = \frac{\sum p_t \sqrt{q_0 q_0^D}}{\sum p_0 \sqrt{q_0 q_0^D}}$	$P_{0t}^{GAH}(0,\tfrac{1}{2}) = \frac{\sum p_t \sqrt{q_0 q_0^I}}{\sum p_0 \sqrt{q_0 q_0^I}}$
j = t	$P_{0t}^{GAH}(t,-\tfrac{1}{2}) = \frac{\sum p_t \sqrt{q_t q_t^I}}{\sum p_0 \sqrt{q_t q_t^I}}$	$P_{0t}^{GAH}(t,\tfrac{1}{2}) = \frac{\sum p_t \left(p_t \sqrt{q_t q_t^I} / p_0\right)}{\sum p_0 \left(p_t \sqrt{q_t q_t^I} / p_0\right)}$

Evidently $P_{0t}^{GAH}(t,-\tfrac{1}{2})$ is the "time antithesis" of $P_{0t}^{GAH}(0,-\tfrac{1}{2})$ because $P_{t0}^{GAH}(t,-\tfrac{1}{2})$ is equal to $\left(P_{0t}^{GAH}(0,-\tfrac{1}{2})\right)^{-1}$. As mentioned above there are some interesting relationships between types of means.

Table 3 (combining tables 1, 1a, and 2)

	I (p_0q_0) j = 0	II (p_0q_t)	III (p_tq_0)	IV (p_tq_t) j = t
k = 0 arithmetic mean	P_{0t}^{L}	P_{0t}^{P}	P_{0t}^{AH}	P_{0t}^{PA}
k = -1 harmonic mean	P_{0t}^{HB}	P_{0t}^{HH}	P_{0t}^{L}	P_{0t}^{P}
k = +1 antiharmonic mean	P_{0t}^{AH}	P_{0t}^{PA}	$P_{0t}^{\bar{H}3}$	$P_{0t}^{GAH}(t,1)$
k = -2 GAH mean	$P_{0t}^{GAH}(0,-2)$	P_{0t}^{RF}	P_{0t}^{HB}	P_{0t}^{HH}

The four white fields represent indices of little or no use. We presented above the relevant formulas for three of them. The remaining formula (RF) reads as follows

$P_{0t}^{RF} = \frac{\sum \left(p_0 q_t^D\right)}{\sum \frac{p_0}{p_t}\left(p_0 q_t^D\right)}$. Note that each of the six (more or less) meaningful index functions of table 1 appears twice in the green fields. We now can offer a list of indices and their "time antitheses". Table 4 shows that we have six pairs:

Table 4

$P_{0t}^{L} \leftrightarrow P_{0t}^{P}$	$P_{0t}^{HB} \leftrightarrow P_{0t}^{PA}$	$P_{0t}^{HH} \leftrightarrow P_{0t}^{AH}$	$P_{0t}^{\bar{H}3} \leftrightarrow P_{0t}^{RF}$	cp. table 7
$P_{0t}^{GAH}(0,-2) \leftrightarrow P_{0t}^{GAH}(t,1) = P_{0t}^{\bar{H}4}$		$P_{0t}^{GAH}(t,-\tfrac{1}{2}) \leftrightarrow P_{0t}^{GAH}(0,-\tfrac{1}{2})$		

From the above formula for P_{0t}^{AH} follows $P_{0t}^{AH} \geq P_{0t}^{L}$ and $P_{0t}^{HH} \leq P_{0t}^{P}$,[14] and as P_{0t}^{HH} is the "time antithesis" of P_{0t}^{AH} the index $\sqrt{P_{0t}^{AH}P_{0t}^{HH}}$ as well as $\sqrt{P_{0t}^{PA}P_{0t}^{HB}}$ follow the model of Fisher's ideal index $P_{0t}^{F} = \sqrt{P_{0t}^{L}P_{0t}^{P}}$. About this more below.[15]

We now are also able to classify unweighted – i.e. weights of s = 1/n – price index formulas using the unweighted GAH formula $\sum\left(\frac{p_t}{p_0}\right)^{k+1} \Big/ \sum\left(\frac{p_t}{p_0}\right)^{k}$ for various values of k

Table 5

k	- 1	- ½	0	+ ½	1
formula	harmonic P^H	Young P^Y using (1.4)	Carli P^C	$\sum \frac{p_t}{p_0} \frac{\sqrt{p_t/p_0}}{\sum\sqrt{p_t/p_0}}$	$\sum \frac{p_t}{p_0} \frac{p_t/p_0}{\sum p_t/p_0}$

The Fisher like index (geometric mean of an index and its time antithesis) is now is the CSWD index $(P^CP^H)^{1/2}$ (as the unweighted analogon to P^F).

2.2. Power means (generalized means) and products of power means

Power means (PM) of order r weighted with expenditure shares s allow to construct new index functions and to demonstrate interesting relationships between existing index functions. The r[th] PM of price relatives with expenditure shares $s_{i0} = p_{i0}q_{i0}/\Sigma\, p_{i0}q_{i0}$ is $P_{0t}^{PM}(r,s_0)$ is given by $P_{0t}^{PM}(r,s_0) = \left[\sum_i s_{i0}\left(\frac{p_{it}}{p_{i0}}\right)^{r/2}\right]^{1/r}$. For r = 2 we have $P_{0t}^{PM}(2,s_0) = \left[\sum_i s_{i0}\left(\frac{p_{it}}{p_{i0}}\right)^{2/2}\right]^{1/2} = \sqrt{P_{0t}^{L}}$,

and $P_{0t}^{PM}(r,s_t)$ for r = -2 (weights $s_{it} = p_{it}q_{it}/\Sigma p_{it}q_{it}$) is $= P_{0t}^{PM}(-2,s_t) = \left[\sum_i s_{it}\left(\frac{p_{it}}{p_{i0}}\right)^{-2/2}\right]^{-1/2} =$ $\sqrt{P_{0t}^{P}}$. This gives rise to study *products of power means*,

$$P_{0t}^{PM}(r,s_0)\, P_{0t}^{PM}(-r,s_t) = \left[\sum_i s_{i0}\left(\frac{p_{it}}{p_{i0}}\right)^{r/2}\right]^{1/r} \cdot \left[\sum_i s_{it}\left(\frac{p_{it}}{p_{i0}}\right)^{-r/2}\right]^{-1/r}, \tag{2.13}$$

of which – as demonstrated – Fisher's ideal index $\sqrt{P_{0t}^{L}P_{0t}^{P}}$ is the special case r = 2.[16] And r= -2 gives $P_{0t}^{PM}(-2,s_0)\, P_{0t}^{PM}(-(-2),s_t) = \sqrt{P_{0t}^{HB}P_{0t}^{PA}}$. Another case is r = 1

$P_{0t}^{PM}(-1,s_t)\, P_{0t}^{PM}(1,s_0) = \sqrt{\frac{\sum p_tq_t}{\sum p_0q_0}\frac{\sum q_0\sqrt{p_0p_t}}{\sum q_t\sqrt{p_0p_t}}} = \sqrt{V_{0t}/Q_{0t}^{W}} = \tilde{P}_{0t}^{W}$ that is the "indirect" Walsh price index (ratio of value index and Walsh quantity index).[17]

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[14] To exceed P^L ($P^{AH} > P^L$) or fall short of P^P is of course not attractive. By contrast to be smaller than Laspeyres makes P^{HB} attractive, and accordingly $P^{PA} > P^P$ may be seen as an advantage of Palgrave's index.
[15] It turns out that most of the Fisher-type indices formed with one of the eight pairs above are nonsensical.
[16] The unweighted variant of this with $s_{it} = s_{i0} = 1/n$ for all i is of course the CSWD index.
[17] That is the product of ❷ and ❶ in table 6.

Table 6

	$A = P_{0t}^{PM}(r,s_0)$	$B = P_{0t}^{PM}(-r,s_t)$	$B^* = P_{0t}^{PM}(r,s_t)$	$A^* = P_{0t}^{PM}(-r,s_0)$
	$\left[\sum_i s_{i0}\left(\frac{p_{it}}{p_{i0}}\right)^{r/2}\right]^{1/r}$	$\left[\sum_i s_{it}\left(\frac{p_{it}}{p_{i0}}\right)^{-r/2}\right]^{-1/r}$	$\left[\sum_i s_{it}\left(\frac{p_{it}}{p_{i0}}\right)^{r/2}\right]^{1/r}$	$\left[\sum_i s_{i0}\left(\frac{p_{it}}{p_{i0}}\right)^{-r/2}\right]^{-1/r}$
1	❶ $\sum q_0\sqrt{p_0p_t}\big/\sum q_0p_0$	❷ $\sum q_tp_t\big/\sum q_t\sqrt{p_0p_t}$	❻	❺
2	❸ $\sqrt{P_{0t}^{L}}$	❹ $\sqrt{P_{0t}^{P}}$	❽	❼
-1	❺ $\sum p_tq_0^D\big/\sum p_0\sqrt{q_0q_0^D}$	❻ $\sum p_t\sqrt{q_tq_t^I}\big/\sum p_oq_t^I$	❷	❶
-2	❼ $\sqrt{P_{0t}^{HB}}$	❽ $\sqrt{P_{0t}^{PA}}$	❹	❸

Obviously B is the "time antithesis" of A (and B* of A*) so that AB or A*B* is a Fisher-type index. Note also that ❺ and ❻ cannot be viewed as price indices because prices p_t in the numerator and p_0 in the denominator are not multiplied by the same factor. The same is true for ❶ and ❷.

2.3. Index type: price and quantity index

The convention is that a price index is a ratio in which numerator and denominator differ with respect to prices but not quantities. The opposite applies to a quantity index: numerator and denominator are different regarding quantities, not prices. However, the divide between a price index on the one hand and a quantity index on the other becomes a bit blurred once we introduce fictitious (deflated or inflated) quantities

Table 7

Price index (in price index formula)		in a quantity index type presentation	
Laspeyres type	Paasche type	Laspeyres type	Paasche type
$P_{0t}^{HB} = \frac{\sum p_tq_0^D}{\sum p_0q_0^D}$	$P_{0t}^{PA} = \frac{\sum p_tq_t^I}{\sum p_0q_t^I}$	$P_{0t}^{HB} = \frac{\sum q_0p_0}{\sum q_0^Dp_0}$	$P_{0t}^{PA} = \frac{\sum q_t^Ip_t}{\sum q_tp_t}$
$P_{0t}^{HH} = \frac{\sum p_tq_t^D}{\sum p_0q_t^D}$	$P_{0t}^{AH} = \frac{\sum p_tq_0^I}{\sum p_0q_0^I}$	$P_{0t}^{HH} = \frac{\sum q_tp_0}{\sum q_t^Dp_0}$	$P_{0t}^{AH} = \frac{\sum q_0^Ip_t}{\sum q_0p_t}$
$P_{0t}^{RF} = \frac{\sum p_t(q_t^Dp_0/p_t)}{\sum p_0(q_t^Dp_0/p_t)}$	$P_{0t}^{\bar{H}3} = \frac{\sum p_t(q_0^Ip_t/p_0)}{\sum p_0(q_0^Ip_t/p_0)}$	$P_{0t}^{RF} = \frac{\sum q_t(p_0^2/p_t)}{\sum q_t^D(p_0^2/p_t)}$	$P_{0t}^{\bar{H}3} = \frac{\sum q_0^I(p_t^2/p_0)}{\sum q_0(p_t^2/p_0)}$

We run into difficulties when we try such a double form in the case of P^L, P^P and the P^{GAH} indices

As to the Fisher type indices we have $P_{0t}^{F} = \sqrt{P_{0t}^{L}P_{0t}^{P}} = \sqrt{\frac{\sum p_tq_0}{\sum p_0q_0}\frac{\sum p_tq_t}{\sum p_0q_t}}$,

$$\sqrt{P_{0t}^{HB}P_{0t}^{PA}} = \sqrt{\frac{\sum p_tq_0^D}{\sum p_0q_0^D}\frac{\sum p_tq_t^I}{\sum p_0q_t^I}} \text{ and } \sqrt{P_{0t}^{AH}P_{0t}^{HH}} = \sqrt{\frac{\sum p_tq_0^I}{\sum p_0q_0^I}\frac{\sum p_tq_t^D}{\sum p_0q_t^D}}.$$

this amounts to combining an index $\leq P^P$ or P^L and one $\geq P^P$ or P^L because

$P^{HH} \leq P^P \leq P^{PA}$ and

$P^{HB} \leq P^L \leq P^{AH}$

3. Average of ratios formulas (AOR), part II: More about the weighted harmonic mean index (P^{HB}) und Palgrave's index (P^{PA})

3.1. Price indices with constant expenditures

The following interpretation of P^{HB} an P^{PA} was brought into play in Germany by Werner Neubauer[18]: The rationale of the two rather unfamiliar indices P^{HB} and P^{PA} can be explained

[18] I (and possibly Neubauer too) did not know that much of the following was seen already by W. Ferger.

using the fictitious quantities introduced in table 1b. The quantity q_0^D that is $q_{i0}^D = q_{i0}(p_{i0}/p_{it})$ in (2.12) $P_{0t}^{HB} = \sum p_0 q_0 / \sum p_0 q_0^D$ must be regarded as quantity you can afford under the regime of new prices p_{it} (q_0^D applies to period t) when you keep your *expenditure* (instead of the quantity) of period 0 constant for each commodity i, so that $q_0^D p_{it} = p_{i0} q_{i0}$ and therefore $\sum q_0^D p_{it} = \sum p_{i0} q_{i0}$. The meaning of inflation now is: the price level is rising to the extent to which we get less quantity for the same amount of money. With rising prices $\lambda_{it} = p_{it}/p_{i0} > 1$ follows from $q_{i0}^D p_{i0} = p_{i0} q_{i0}/\lambda_{it}$ so that $q_{i0}^D < q_{i0}$ holds for *each* commodity, and $\sum p_{i0} q_0^D$ < $\sum p_{i0} q_{i0}$ for all commodities (not to confound with $\sum p_{it} q_0^D = \sum p_{i0} q_{i0}$).

To interpret the products $p_t q_0^D$ and $p_0 q_t^I$ is easy (as they are equal to $p_0 q_0$ and $p_t q_t$ respectively), but $p_0 q_0^D$ and $p_t q_t^I$ does not seem to be meaningful. However, with $\lambda > 1$ one can conclude

Table 8

fictitious vs. real quantities	fictitious vs. real expenditures (interpretation)	price index	the underlying perspective of the interpretation
q_0^D refers to t $\Rightarrow q_0^D < q_0$	$\sum p_0 q_0^D < \sum p_0 q_0$ had I bought in 0 (prices p_0) the smaller quantity q_0^D (the one I can still afford in t) it had cost less than $\Sigma p_0 q_0$.	$P_{0t}^{HB} > 1$	to know and make use of q_0^D in t so that $\sum p_{it} q_0^D = \Sigma p_{i0} q_{i0}$ requires to **remember** in t the expenditure of 0 ($\Sigma p_{i0} q_{i0}$)
q_t^I refers to 0 $\Rightarrow q_t^I > q_t$	$\sum p_t q_t^I > \sum p_t q_t$ if I would buy in t the larger quantity I could have bought in 0 (with prices $p_0 < p_t$) it would cost more now, in t	$P_{0t}^{PA} > 1$	to know and make use of q_t^I in 0 so that $\sum p_{i0} q_{it}^I = \Sigma p_{it} q_{it}$ requires to know in 0 the **future** expenditure of t ($\Sigma p_{it} q_{it}$)

To remember in t what we used to spend in 0 in order to adjust q_0 so that q_0^D $\sum q_0^D p_{it} = \sum p_{i0} q_{i0}$ is clearly a more realistic perspective than to transform q_0 to q_t^I so that we now (in 0) with current prices p_0 spend the same amount of money we are going to spend in the future (with prices p_t and quantities q_t). So we have again the situation that the implicit thought experiment with P^{PA} is more doubtful that in the case of P^{HB} just like the underlying logic of P^P is not on an equal footing with P^L (as is generally believed) but clearly less stringent.

3.2. Price indices as reciprocal quantity indices

It is interesting to see what becomes apparent when we contrast P^{HB} and P^{PA} to reciprocal quantity indices of Laspeyres and Paasche:

Table 9

	price	reciprocal quantity
HB and Laspeyres	$P_{0t}^{HB} = \frac{\sum p_0 q_0}{\sum p_0 q_0^D}$	$\left(Q_{0t}^L\right)^{-1} = \frac{\sum p_0 q_0}{\sum p_0 q_t}$
PA and Paasche	$P_{0t}^{PA} = \frac{\sum p_t q_t^I}{\sum p_t q_t}$	$\left(Q_{0t}^P\right)^{-1} = \frac{\sum p_t q_0}{\sum p_t q_t}$

Numerator and denominator differ in P^{HB} by $q_{i0}^D \neq q_{i0}$ and in P^{PA} by $q_{it}^I \neq q_{it}$ only. So P^{HB} and P^{PA} reflect the development of quantities.

Moreover for all commodities holds $\frac{q_{i0}^D / q_{i0}}{p_{it}/p_{i0}} = 1$ and $\frac{q_{it}^I / q_{it}}{p_{it}/p_{i0}} = 1$.

Hence P_{0t}^{HB} and P_{0t}^{PA} may also be viewed as a reciprocal quantity indices using fictitious quantities. The ratios on $\frac{q_{i0}^{D}/q_{i0}}{p_{it}/p_{i0}} = \frac{q_{i0}^{D}/p_{it}}{q_{i0}/p_{i0}} = 1$ and equivalently $\frac{q_{it}^{I}/q_{it}}{p_{it}/p_{i0}} = 1$ may be viewed as (ratios) of elasticities. The "inverse quantity index" interpretation of a price index P^{HB} or P^{PA} is in line with the concept of inflation (as mentioned above) according to which a price level is rising to the extent to which we get less quantity for the same amount of money. It is also in harmony with reciprocal prices as fictitious quantities in the early index theory.

Not only P^{HB} and P^{PA}, also the Laspeyres and Paasche price index can be given an appearance of a reciprocal quantity index, because we have $P_{0t}^{L} = \sum p_t q_0 \big/ \sum p_t q_0^{D}$ and $P_{0t}^{P} = \sum p_0 q_t^{I} \big/ \sum p_0 q_t$ as counterparts of $P_{0t}^{HB} = \sum p_0 q_0 \big/ \sum p_0 q_0^{D}$ and $P_{0t}^{PA} = \sum p_t q_t^{I} \big/ \sum p_t q_t$.

In Ferger 1931 a correctly developed theory can be found about when to use the weighted (weights w_i where $w = \Sigma w_i$) arithmetic mean $\bar{x} = \frac{1}{w}\sum_i x_i w_i$ as opposed to the harmonic mean $(\bar{x}_H)^{-1} = \frac{1}{w}\sum_i \frac{1}{x_i} w_i$. Ferger realized that $\bar{x}_H$ should be used when the task is to average ratios $x_i = \frac{n_i}{d_i}$ where the numerator is constant ($n_i = n\ \forall i$) while $\bar{x}$ is the correct average, when the denominator is constant ($d_i = d\ \forall i$) so that x_i is a linear transformation of n_i.[19] Prices are by their very nature always ratios[20] $p_i = t_i/m_i$ = expenditure for good i (monetary transaction)/mass (quantity of good i). So a household might be aiming to procure a constant (given) quantity, say $m = q_0$ (to keep base period q_0 constant (in which case P^L would be appropriate) or to maintain the (base period) expenditure constant.[21] And to our surprise on p. 40 we found the formula of "our" index P_{0t}^{HB} derived as weighted harmonic mean $P_{0t}^{HB} = \left(\sum \frac{p_0}{p_t} \frac{p_0 q_0}{\sum p_0 q_0} \right)^{-1}$.

We will come back to Ferger and thus to another interpretation of P^{HB} in a short annex below.

3.3. Again Laspeyres (P^L) and Paasche (P^P) are not equally well reasoned indices[22]

Hence P^L is related to P^{HB} and so is P^P to P^{PA}. However, as noted already in sec. 3.1 to interpret P^{PA} seems less straightforward than P^{HB}, because q_{it}^{I} appears more fictitious than q_{i0}^{D}. To argue in terms of q_t^{I} (based on future quantities q_{it}) for a fictitious base period expenditure (as in P^{PA}) is much less plausible, than to introduce quantities q_{i0}^{D} for the future based on quanti-

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[19] The "classic" example is velocities $v_i = D_i/T_i$ = distance/time (i = 1,…,m) of m bikers. When all m bikers have the same distanced to travel, so that they differ with respect to their elapsed time $T_1, \ldots, T_m$ the harmonic mean $\bar{v}_H$ is to be taken; while the arithmetic mean $\bar{v}$ would be correct when the winner is the one who covered the longest distance in a given time T. So a ratio can be viewed from different perspectives, and in P^L and P^{HB} simply prices are viewed from different perspectives, both reasonable and legitimate.

[20] This point was also (and at great length) made by Allyn Young 1921. We use the symbols t and m Young introduced, t_i instead of $p_i q_i$ for the expenditure and m_i instead of q_i for the quantity (and t rather than q_0).

[21] Ferger discussed in detail which approach (orientation at quantities or at expenditures) would reflect more realistically the decisions of households, and he found that constancy of quantities q_0 should be ruled out because it would imply a zero price elasticity of demand. He explicitly derived P^{HB} which he recommended for use, but he did not discuss how P^{HB} is related to P^L and that P^L would be a kind of arithmetic analogon to the harmonic P^{HB}. Interestingly and most noteworthy: he already saw that P^{HB} vs. P^L boils down to constant expenditures vs. constant quantities. So he already was aware of an interpretation Neubauer some thirty years later rediscovered.

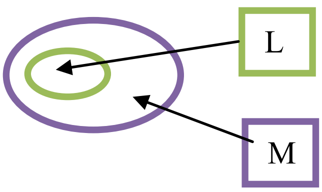
[22] Or put more precisely: Laspeyres is the better reasoned measure of inflation than Paasche.

ties q_{i0} of the past (as in P^{HB}). It makes sense to ask: what can we consume in the future in order to keep to the expenditure of the past period 0, however, it is much less sensible to ask what we could have consumed in the past when we spent the same amount of money we do now, that is in the future period t (aiming at spending or imagining an expenditure amounting to $p_{i0}q^{I}_{it} = p_{it}q_{it}$ *in the past* as in P^{P} or in P^{PA} – where P^{PA} is even less rational because in P^{PA} we recourse in a fictitious quantity q^{I}_{t} - is less realistic a problem than spending *now, in t* the sum $q^{D}_{0}\, p_t$ as we actually did in the past, when we spent q_0p_0). So we have here with P^{PA} and P^{HB} the same asymmetric relation as with P^{P} and P^{L} which shows once more, that P^{P} and P^{L} are – contrary to a common preoccupation – not equally reasonable index functions. There is a significant difference between these two indices that can easily be seen when one compares the sequence $P^{L}_{01} = \frac{\sum p_1q_0}{\sum p_0q_0}$, $P^{L}_{02} = \frac{\sum p_2q_0}{\sum p_0q_0}$, $P^{L}_{03} = \frac{\sum p_3q_0}{\sum p_0q_0}$,... where subsequent indices differ only with respect to prices in the numerator, however in $P^{P}_{01} = \frac{\sum p_1q_1}{\sum p_0q_1}$, $P^{P}_{02} = \frac{\sum p_2q_2}{\sum p_0q_2}$, $P^{P}_{03} = \frac{\sum p_3q_3}{\sum p_0q_3}$,... also with constantly changing quantities in numerator and denominator so that the elements P_{01}, P_{02}, P_{03}, … in the sequence of prices indices are not comparable among themselves (what evidently applies to P^{HB} and P^{PA} too).

3.4. Axiomatic performance of P^{HB} and P^{PA}

Due to the relationship between arithmetic and harmonic means $P^{HB}_{0t} < P^{L}_{0t}$ and $P^{PA}_{0t} > P^{P}_{0t}$. This may be welcomed as a valuable property of P^{HB} and P^{PA}, because it is often said that Paasche understates and Laspeyres overstates inflation as measured by a true price index: the interval $P^{PA}_{0t} < P^{TRUE}_{0t} < P^{HB}_{0t}$ is smaller than $P^{P}_{0t} < P^{TRUE}_{0t} < P^{L}_{0t}$. In this sense it also might be somewhat attractive to consider the time reversible price index $\sqrt{P^{HB}_{0t}P^{PA}_{0t}}$, built analogously to $P^{F}_{0t} = \sqrt{P^{L}_{0t}P^{P}_{0t}}$ although an economic interpretation of this index must be challenging.

Both indices, P^{HB} and P^{PA} are weakly monotonous, however, P^{PA} is *strictly* monotonous only in the base period prices and P^{HB} in the current period prices. Though all indices of table 1 are means of price relatives, that is they possess the mean value property M, only few of them, viz P^{L} and P^{P} are linear (= additive): Additivity (linearity) L is defined in table 10b and neither P^{PA} nor P^{HB} satisfy both conditions of additivity.[23] Property L is a subset of property M



Linearity (L) is a valuable property because non-linearity (non-additivity) means that it cannot be simply inferred how the index will change in response to the change of individual prices (denoted by non-zero vectors $\mathbf{p}^{+}_{t}$ or $\mathbf{p}^{+}_{0}$ respectively). P^{PA}_{0t} is not additive in current period prices and P^{HB}_{0t} not additive (linear) in base period prices. In other words, there is no strict relationship between the change of individual prices $\Delta p_{it} = p^*_{it} - p_{it} = p^{+}_{it}$ and the amount by which the index as a whole changes $\Delta P = P(\mathbf{p}^{+}) = P(\mathbf{p}^*) - P(\mathbf{p})$.

[23] Other examples of price indices which clearly are means of price relatives but nonetheless not linear are the logarithmic Laspeyres (also called geometric Laspeyres) price index DP^{L} where $\ln(DP^{L})$ is the arithmetic mean of logarithmic price relatives (weighted with expenditure shares of the base period $p_0q_0/\Sigma p_0q_0$), or the quadratic mean of price relatives P^{QM}.

Palgrave's index $P_{0t}^{PA} = \sum \frac{p_t}{p_0} p_t q_t \Big/ \sum p_t q_t$ is not linear (additive) in current period prices because this would require for $\mathbf{p}_t^* = \mathbf{p}_t + \mathbf{p}_t^+$ to hold that

$P_{0t}^{PA}(\mathbf{p}_t^*) = \left(\sum \frac{p_t q_t}{p_0} p_t + \sum \frac{p_t q_t}{p_0} p_t^+ + \sum \frac{p_t^+ q_t}{p_0} p_t + \sum \frac{p_t^+ q_t}{p_0} p_t^+ \right) \Big/ \left(\sum p_t q_t + \sum p_t^+ q_t \right)$ is equal to the sum of $P_{0t}^{PA}(\mathbf{p}_t) = \sum \frac{p_t q_t}{p_0} p_t \Big/ \sum p_t q_t$ and $P_{0t}^{PA}(\mathbf{p}_t^+) = \sum \frac{p_t^+ q_t}{p_0} p_t^+ \Big/ \sum p_t^+ q_t$.

Table 10(a)

	price index	
	P_{0t}^{HB} (harmonic base)	P_{0t}^{PA} (Palgrave)
bounds	$P_{0t}^{HB} < P_{0t}^{L}$	$P_{0t}^{P} < P_{0t}^{PA}$
monotonicity*	*weak* monotonicity fulfilled by both indices, P^{HB} and P^{PA} P^{PA} *strictly* monotonous only in the base period prices, P^{HB} is *strictly* monotonous only in current period prices	
additivity	not in base period prices	not in current period prices
aggregative consistency (AC)	both indices, P_{0t}^{HB} and P_{0t}^{PA} are aggregative consistent	
quantity indices	both indices fail the factor reversal test and the indirect (implicit) quantity indices violate the proportionality axiom $\widetilde{Q}_{0t} \neq \lambda$ although $q_{it} = \lambda q_{i0}$ $\forall i$	

* the reason for failing monotonicity is that $p_t^* > p_t$ affects in P^{PA} both, price relatives *and* weights (in P^{HB}, however, only price relatives). The opposite applies to P^{HB} that meets strict monotonicity only in current period prices but not in base period prices).

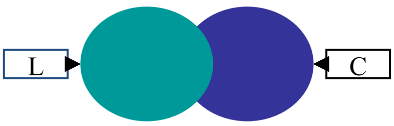
Table 10(b)

Axiom	Definition
weak monotonicity	$P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t) > P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_0,\mathbf{q}_t)$ if $\mathbf{p}_t > \mathbf{p}_0$ $P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t) < P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_0,\mathbf{q}_t)$ if $\mathbf{p}_t < \mathbf{p}_0$
strict monotonicity	a) in current period prices $P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t^*,\mathbf{q}_t) > P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t)$ if $\mathbf{p}_t^* \geq \mathbf{p}_t$ b) in base period prices $P(\mathbf{p}_0^*,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t) < P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t)$ if $\mathbf{p}_0^* \geq \mathbf{p}_0$
additivity (= linearity)	a) in current period prices $P(\mathbf{p}_0,\mathbf{p}_t^*) = P(\mathbf{p}_0,\mathbf{p}_t) + P(\mathbf{p}_0,\mathbf{p}_t^+) = A + B$ if $\mathbf{p}_t^* = \mathbf{p}_t + \mathbf{p}_t^+$ (can be violated by P^{PA}) b) in base period prices $\frac{1}{P(\mathbf{p}_0^*,\mathbf{p}_t)} = \frac{1}{P(\mathbf{p}_0,\mathbf{p}_t)} + \frac{1}{P(\mathbf{p}_0^+,\mathbf{p}_t)} = \frac{1}{C} + \frac{1}{D}$ if $\mathbf{p}_0^* = \mathbf{p}_0 + \mathbf{p}_0^+$. (can be violated by P^{HB})
aggregative consistency (AC)	compilation of an index for the total aggregate (T) *in one step* starting with individual price relatives (r) in symbols $r_1, r_2, \ldots, r_n \Rightarrow P_T$ or *in two steps* via indices for sub-aggregates (S1, S2, …) or again in symbols: $r_1, r_2, \ldots, r_n \Rightarrow P_{S1}, P_{S2}, \ldots \Rightarrow P_T$ will yield the same result

It can easily be seen that P^{PA} not even complies with linearity in the restricted case of $\mathbf{p}_t^+ = \mathbf{b} = \begin{bmatrix} b \\ \dots \\ b \end{bmatrix}$ which means that "if all prices are increased by the same amount, the index of the new prices should equal to the old index number plus the index number of the constant amount".[24] A price index formula that complies with additivity (linearity) in *both*, base period *and* current period prices can also be written as $P_{0t} = \frac{\mathbf{a}'\mathbf{p}_t}{\mathbf{b}'\mathbf{p}_0}$, that is as ratio of scalar vector products.[25] For example P_{0t}^L implies $\mathbf{a}' = \mathbf{b}' = \mathbf{q}_0'$ [constant], in P_{0t}^P we have $\mathbf{a}' = \mathbf{b}' = \mathbf{q}_t'$. But P_{0t}^{HB} and P_{0t}^{PA} though both means of price relatives are yet not additive (linear) functions in prices p_0 *and* p_t. Though we can write (see table 7) $P_{0t}^{HB} = \frac{\sum p_t q_0^D}{\sum p_0 q_0^D}$ and $P_{0t}^{PA} = \frac{\sum p_t q_t^I}{\sum p_0 q_t^I}$ this is not in line with $P_{0t} = \mathbf{a}'\mathbf{p}_t / \mathbf{b}'\mathbf{p}_0$ because unlike **a** and **b** the "quantity" vectors here (with elements $q_t^I = q_t(p_t/p_0)$ and $q_0^D = q_0(p_0/p_t)$ respectively) are not independent of $\mathbf{p}_t$ and $\mathbf{p}_0$.

> *Aside:* The presentation in table 7 is particularly useful to relate these indices to linear indices. However the forms $P_{0t}^{PA} = \sum p_t q_t^I / \sum p_t q_t$ and $P_{0t}^{HB} = \sum p_0 q_0 / \sum p_0 q_0^D$ are better in order to demonstrate that under normal conditions of rising prices the indices exceed unity because then $q_t^I > q_t$ and $q_0^D < q_0$.

Just like linearity or additivity (L) and the mean value (of price relatives) indices (M) and should be kept distinct so also additivity (L) aggregative consistency (AC or simply C). However, while L is a subset of M the situation with L and C is different



- $C \cap L$ (for example P^L, P^P),
- C-L (P^{PA},P^{HB}) so that P^{HB} and P^{PA} meet C though not L
- for L-C an example is $\frac{1}{2}(P_{0t}^L + P_{0t}^P)$ which is linear but violates C because the weights ½ are not related to expenditure shares; and finally for
- $\overline{C \cup L}$ the most prominent example is P^F.

In the case of P^{HB} and K = 2 sub-aggregates AC means that when the index for the total aggregate can be compiled in one step, say as harmonic mean of price relatives, the corresponding two step procedure via compiling indices for K sub-aggregates which are subsequently aggregated (again as harmonic mean of the K sub-indices with base year expenditure weights) to the total index we should get the same result. It can easily be seen that indeed for K = 2 sub-aggregates, A and B, the total index P_{0t}^{HB} is a harmonic mean of $P_{0t}^{HB}(A)$ and $P_{0t}^{HB}(B)$

$$\left(P_{0t}^{HB}\right)^{-1} = \left(\sum \frac{p_{A0}}{p_{At}} \frac{p_{A0}q_{A0}}{\sum p_{A0}q_{A0}}\right) \frac{\sum p_{A0}q_{A0}}{\sum p_0 q_0} + \left(\sum \frac{p_{B0}}{p_{Bt}} \frac{p_{B0}q_{B0}}{\sum p_{B0}q_{B0}}\right) \frac{\sum p_{B0}q_{B0}}{\sum p_0 q_0}$$

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[24] Pfouts 1966 p. 176 also noticed that the famous "ideal" index P^F of Fisher does not satisfy additivity (in all variants) either, not even in the restricted case above **b**' = [b b … b]. His paper was a sort of pamphlet against the (unmerited) prestige of the "ideal" Fisher index P^F, an index he thought, should be abandoned.

[25] von der Lippe 2007, 193.

$$=\left(P_{0t}^{HB}(A)\right)^{-1} g_A + \left(P_{0t}^{HB}(B)\right)^{-1} g_B \text{, with } g_A + g_B = \frac{\sum p_{A0}q_{A0}}{\sum p_0 q_0} + \frac{\sum p_{B0}q_{B0}}{\sum p_0 q_0} = 1.$$ [26]

Other important properties of P^{HB} and P^{PA} concern quantity indices of the HB and PA type. A distinction has to be made between

- *direct quantity indices* Q_{0t} gained from price indices P_{0t} by interchanging prices and quantities so that $Q_{0t}^{HB} = \sum p_0 q_0 \Big/ \sum \frac{q_0}{q_t} p_0 q_0$ and $Q_{0t}^{PA} = \sum \frac{q_t}{q_0} p_t q_t \Big/ \sum p_t q_t$, and
- indirect (implicit) quantity indices resulting from dividing the value index $V_{0t} = \Sigma p_t q_t / \Sigma p_0 q_0$ by the respective price index $\tilde{Q}_{0t} = V_{0t}/P_{0t}$.

Unlike the direct quantity indices Q_{0t}^{HB} and Q_{0t}^{PA} the respective implicit quantity indices, that is $\tilde{Q}_{0t}^{HB} = V_{0t}/P_{0t}^{HB}$ and $\tilde{Q}_{0t}^{PA} = V_{0t}/P_{0t}^{PA}$ will violate proportionality in the quantities and consequently also identity, This clearly invalidates them as deflators.[27] The products $A = P_{0t}^{PA} Q_{0t}^{PA}$ and $B = P_{0t}^{HB} Q_{0t}^{HB}$ are in general unequal ($A \neq B$) and also different from the value index V_{0t}, that is the indices of the HB and PA type are not factor reversible.[28] Lack of proportionality of $\tilde{Q}_{0t}^{HB} \neq Q_{0t}^{HB}$ and $\tilde{Q}_{0t}^{PA} \neq Q_{0t}^{PA}$ (and thus a fortiori lack of identity),[29] can easily be verified as follows: if $q_{it} = \lambda q_{i0}\ \forall i$ then also $Q_{0t}^{P} = Q_{0t}^{L} = \lambda$ and therefore the indirect quantity indices will in general be $\tilde{Q}_{0t} \neq \lambda$ because $\tilde{Q}_{0t}^{HB} = V_{0t}/P_{0t}^{HB} = Q_0^P P_{0t}^L / P_{0t}^{HB} = Q_0^L P_{0t}^P / P_{0t}^{HB} \neq \lambda$ unless $P_{0t}^{HB} = P_{0t}^{P} = P_{0t}^{L}$. The same consideration leads to $\tilde{Q}_{0t}^{PA} \neq \lambda$. More specifically we can conclude $\tilde{Q}_{0t}^{HB} > \lambda = \tilde{Q}_{0t}^{P}$ because $P^L > P^{HB}$ and $\tilde{Q}_{0t}^{PA} < \lambda$.

It is difficult to say something about how $\sqrt{P_{0t}^{HB} P_{0t}^{PA}} = \sqrt{\frac{\sum p_t q_t^D}{\sum p_0 q_t^D} \frac{\sum p_t q_t^I}{\sum p_0 q_t^I}}$ differs from $P_{0t}^F = \sqrt{P_{0t}^L P_{0t}^P} = \sqrt{\frac{\sum p_t q_0}{\sum p_0 q_0} \frac{\sum p_t q_t}{\sum p_0 q_t}}$ because it is the structure of weights that matters and nothing definite can be said about how the structure of weights in P^L vs. P^{HB} and in P^P vs. P^{PA} differs:

$$P_{0t}^L = \frac{\sum p_t q_0}{\sum p_0 q_0} = \sum \frac{p_t}{p_0} \frac{p_0 q_0}{\sum p_0 q_0} = A \cdot P_{0t}^{HB} = A \cdot \sum \frac{p_t}{p_0} \frac{p_0 q_0^D}{\sum p_0 q_0^D} \text{ and}$$

$$P_{0t}^P = \sum \frac{p_t}{p_0} \frac{p_0 q_t}{\sum p_0 q_t} = \frac{1}{B} P_{0t}^{PA} = \sum \frac{p_t}{p_0} \frac{p_0 q_t^I}{\sum p_0 q_t^I}.$$ It is difficult make conclusions about A/B.

[26] It can easily be verified that the corresponding equation (with arithmetic means) holds for P^{PA}, so that also Palgrave's index is additively consistent.

[27] When all quantities remain constant ($\lambda = 1$), then using P^{HB} as deflator would not result in $V/P^{HB} = 1$ unless also all prices would remain constant (because then we have $V=P^{HB} = 1$). Otherwise V/P^{HB} shows (counterfactually) a rising volume (and the deflator P^{PA} analogously a decreasing volume).

[28] Factor reversibility (the factor reversal test) amounts to equality of the direct and the indirect quantity index. Unlike the pair L/P (Laspeyres/Paasche), the pair HB/PA does not even pass the less demanding product-test.

[29] Identity is the special case $\lambda = 1$ of proportionality. Hence these indices cannot be regarded as quantity indices in the strict sense.

4. Weighted indices: Ratios of averages formulas (ROA)

Many index functions possess both forms (formulas), an AOR form like $P_{0t}^{L} = \sum \frac{p_{it}}{p_{i0}} \frac{p_{i0}q_{i0}}{p_{i0}q_{i0}}$ with weights $p_{i0}q_{i0} / \sum p_{i0}q_{i0}$ and a ROA form $P_{0t}^{L} = \sum p_{it}q_{i0} / \sum p_{i0}q_{i0}$ in which both numerator and denominator reflect expenditure or average prices if divided by a suitable quantity. We encountered indices with an AOR form and an unfamiliar ROA form with no or only an at best far fetched AOR interpretation.[30] To give only one example for an "ROA-only" index (with no AOR interpretation) we briefly mention Drobisch's index as ratio of a sort of absolute (e.g. expressed in \$) price levels $\widetilde{P}_t$ and $\widetilde{P}_0$ as "macro level" or (universal) "unit values"

(3.1) $\widetilde{P}_t = \frac{\sum p_{it}q_{it}}{\sum q_{it}} = \sum p_{it} \frac{q_{it}}{\sum q_{it}}$ and $\widetilde{P}_0$ defined correspondingly, we get

(3.2) $P_{0t}^{DR} = \frac{\widetilde{P}_t}{\widetilde{P}_0} = \frac{\sum p_{it}q_{it}}{\sum p_{i0}q_{iß}} \cdot \frac{\sum q_{i0}}{\sum q_{it}} = \frac{V_{0t}}{Q_{01}^{D}}$ with Dutot's quantity index $Q_{0t}^{D} = \sum q_{it} / Q_{i0}$.

One of the problems with P_{0t}^{DR} is that the sum over all goods and services is generally not defined (the summation fails already for the simple reason that there are quite different measuring units for the quantities). So as a rule P_{0t}^{DR} cannot be compiled on the level of a national economy. This infeasible P_{0t}^{DR} is often called "unit value index"(UVI) and it is often confounded with another indeed existing (in particular in foreign trade statistics where we have since long the habit to report quantities of exported or imported goods) UVI index, compiled with unit values (a sort of average prices) as building blocs instead of prices as building blocs. Assume the k-th aggregate can be decomposed into J_k individual goods j = 1,2,…,J_k. then its unit value in t is $\widetilde{p}_{kt} = \frac{\sum_j p_{jkt}q_{jkt}}{\sum_j q_{jkt}} = \frac{\sum p_{jkt}q_{jkt}}{q_{kt}}$ and in 0 $\widetilde{p}_{k0} = \frac{\sum p_{jk0}q_{jk0}}{q_{k0}}$,[31] so that a Laspeyres type (L) or (more commonly in use) Paasche (P) type unit value index can be compiled as $P_{0t}^{UV(L)} = \frac{\sum_k \widetilde{p}_{kt}q_{k0}}{\sum_k \widetilde{p}_{k0}q_{k0}} \neq P_{0t}^{L} = \frac{\sum_k \sum_j p_{jkt}q_{jk0}}{\sum_k \sum_j p_{jk0}q_{jk0}}$ and $P_{0t}^{UV(P)} = \frac{\sum_k \widetilde{p}_{kt}q_{kt}}{\sum_k \widetilde{p}_{k0}q_{kt}} \neq P_{0t}^{P} = \frac{\sum_k \sum_j p_{jkt}q_{jkt}}{\sum_k \sum_j p_{jk0}q_{jkt}}$.

Both indices evidently differ from P_{0t}^{DR}.[32] The present author has repeatedly published papers explaining the difference between *unit value* indices $P_{0t}^{UV(L)}$ and $P_{0t}^{UV(P)}$ and genuine *price* indices P_{0t}^{L} and P_{0t}^{P}. There may be other "ROA-only" indices in addition to P_{0t}^{DR}, but it seems questionable that it is worthwhile discussing them. In von der Lippe (2013) we gave mention to a rather strange index formula of Lehr which is another example (in addition to P^{DR}) for an "ROA-only" index.

Appendix: *Ferger's index of purchasing power (PP)*

According to Ferger an appropriate measure of the purchasing power (PP) of money is "not the reciprocal of an index of prices but an *index of the reciprocals of prices* properly

[30] This applies to indices making use of fictitious quantities. There are of course also indices that possess none of the two forms, neither AOR nor ROA, for example Fisher's ideal index P^F.

[31] Unlike the overall sum of quantities $\Sigma_k\Sigma_j q_{jkt}$ and $\Sigma\Sigma q_{jkt}$ the k-specific $q_{kt} = \Sigma_j q_{jkt}$ and q_{k0} can be compiled.

[32] It may be noted in passing that P^{DR} is able to comply with the chain test (transitivity), however, fails (what already Laspeyres noticed) identity because if $p_{i1} = p_{i0}$ $\forall i$ we have $P^{DR} = Q^L/Q^D$.

weighted" Ferger 1936, p 266 (Ferger's emphasis). As an unweighted version of such an index of reciprocal prices P^{rp} he studied $\frac{\frac{1}{n}\sum \frac{1}{p_{it}}}{\frac{1}{n}\sum \frac{1}{p_{i0}}} = \frac{\left(\bar{p}_t^H\right)^{-1}}{\left(\bar{p}_0^H\right)^{-1}} = \frac{\bar{p}_0^H}{\bar{p}_t^H}$, and he demonstrated[33] the difference between his P^{rp} concept and the ruling "macro" PP concept as reciprocal price index (P^{-1}), with the example $\left(P_{0t}^D\right)^{-1} = \frac{\bar{p}_0}{\bar{p}_t}$. The reason why he rejected this P^{-1} approach of inverting a price *index* was, that he opposed the then (in the "stochastic index theory") popular idea of a hidden general driving force that makes all prices increase or decrease more or less in unison.[34] So he thought a measure of PP should address individual prices as a sort of "micro" approach[35] and consider expenditures rather than quantities.[36] Ferger realizeded that the difference between his favoured index $\bar{p}_0^H / \bar{p}_t^H$ and the P^{-1} approach using $\left(P_{0t}^D\right)^{-1} = \bar{p}_0 / \bar{p}_t$

> "arises from the different weights applied by the formula despite our proposed intention of maintaining equal weights. The price index gives more weight to the price of the expensive article A by assuming constant quantities to be bought, while the purchasing power index gives more weight to the price change of the cheaper article B by assuming constant expenditure to be maintained" (268).

For a weighted index of PP (i.e. of reciprocal prices) Ferger considered expenditures $e_0 = p_{i0}q_{i0}$, but such weights need an adjustment because what we get for 1\$ and thus for e_0\$ depends on the prices. So weights should be $\frac{p_{i0}q_{i0}}{\frac{1}{p_{i0}}} = (p_{i0})^2 q_{i0}$ and it can easily be seen that

$\frac{\sum (1/p_{it})(p_{i0})^2 q_{i0}}{\sum (1/p_{i0})(p_{i0})^2 q_{i0}} = \frac{\sum p_0 q_0 (p_0/p_t)}{\sum p_0 q_0} = \frac{1}{P_{0t}^{HB}}$ and $\frac{\sum (1/p_{it})(p_{it})^2 q_{it}}{\sum (1/p_{i0})(p_{it})^2 q_{it}} = \frac{1}{P_{0t}^{PA}}$. Division by $1/p_{i0}$ or $1/p_{it}$ means that expenditure is not measured in \$ but in units[37] per \$ which comes closer to a sort of standardized and dimensionless quantity.

As an aside: For Ferger's ***PP*** index $(P^{HB})^{-1}$ but not for the ***price*** index P^{HB} we have a time series interpretation as we had in sec. 3.3 (p. 12 above) for the price index P^L of Laspeyres because of the constant denominator $\frac{\sum p_0 q_0 (p_0/p_1)}{\sum p_0 q_0}, \frac{\sum p_0 q_0 (p_0/p_2)}{\sum p_0 q_0}, \frac{\sum p_0 q_0 (p_0/p_3)}{\sum p_0 q_0}, \ldots$ In this respect the PP index $(P^{PA})^{-1}$ is less attractive as $(P^{HB})^{-1}$ just like P^{PA} is short of P^{HB} (or P^P short of P^L). Interestingly Ferger started out with the P^{rp} idea but ended up with a P^{-1} type of

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[33] Using a numerical example, p. 267.

[34] He also disapproved the preference (in the stochastic index theory) for unweighted indices which was based on the idea that each price relative was considered an independent representative of the same general force driving prices up or down. So Ferger was explicitly in favour of weighting with expenditure weights (we already noticed above his interesting interpretation of P^{HB} in terms of keeping *expenditures* rather than *quantities* constant).

[35] This is of course rather absurd: we cannot speak of a purchasing power of money with respect to cheese, shoes etc. but only of a purchasing power of a specified income spent for a *bundle* of clearly defined purchases. Notwithstanding his idea "There is no change in the value of money except that which is result of, or rather is composed of changes in the prices of commodities"(262f) is of course correct.

[36] "If we are measuring changes in its [the money's] buying power over several commodities, we must thus measure the changing quantities of these commodities when the same amounts of money are paid for them respectively, rather than measuring the varying total cost of the same respective quantities" (Ferger 1936, p. 268).

[37] Interestingly Fergers symbol for $1/p_0$ is u_0 (u for "unit"). We already saw that $1/p_0$ or $1/p_t$ may be viewed as "implicit" quantities. And Ferger indeed made extensive reference to Bowley who stressed that PP is closely related to a quantity and so a PP index should be somehow conceptually related to a quantity index.

PP index, viz. $(P^{HB})^{-1}$. So in practice his distinction by which he set great store that is "not the reciprocal of an index of prices but an *index of the reciprocals of prices*" does not seem to be a particularly fertile one.

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