Relations between linear price indices

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1. Laspeyres and Paasche index; a theorem of v. Bortkiewicz

Ladislaus von Bortkiewicz (1868 – 1931) developed in 1923 a well known theorem according to which the difference between the price index of Laspeyres (P_{01}^L) and Paasche (P_{01}^P) depends on the (weighted) covariance between price relatives (p_{1i}/p_{0i}) and quantity relatives (q_{1i}/q_{0i}) of n commodities. The covariance is given by

$$(1) \qquad c_{xy} = \sum_i \left(\frac{p_{i1}}{p_{i0}} - P_{01}^L\right)\left(\frac{q_{i1}}{q_{i0}} - Q_{01}^L\right)\frac{p_{i0}q_{i0}}{\sum p_{i0}q_{i0}} = V_{01} - Q_{01}^L P_{01}^L = Q_{01}^L P_{01}^P - Q_{01}^L P_{01}^L$$

where V_{01} is the value ratio (or value index) $V_{01} = \Sigma p_1q_1/\Sigma p_0q_0$ (we drop the subscript i because it is clear that summation takes place over n commodities) and P_{01}^L, Q_{01}^L are price and quantity indices respectively of Laspeyres, while P_{01}^P, Q_{01}^P are those indices according to the Paasche formula. As by definition $V_{01} = Q_{01}^P P_{01}^L = Q_{01}^L P_{01}^P$ holds we get.

$$(1a) \qquad \bar{c}_{xy} = \frac{c_{xy}}{P_{01}^L Q_{01}^L} = \frac{P_{01}^P}{P_{01}^L} - 1 = \frac{Q_{01}^P}{Q_{01}^L} - 1$$

where $\bar{c}_{xy}$ is the *centred* covariance. The important result now is

- $P_{01}^P < P_{01}^L$ (or equivalently $Q_{01}^P < Q_{01}^L$) when the (centred) covariance c_{xy} and thus also the (centred) covariance $\bar{c}_{xy}$ is negative and conversely (which, however, is the less frequent case empirically)
- $P_{01}^P > P_{01}^L$ ($Q_{01}^P > Q_{01}^L$) when price- and quantity relatives are positively correlated, that is c_{xy} is positive (thus also $\bar{c}_{xy} > 0$).[1]

2. The generalized theorem of Bortkiewicz

It can easily be seen that this result is simply a special cases of a more general theorem on two *linear* indices (ratios of scalar products of vectors **x** and **y**).[2] Assume

$$(2) \qquad X_0 = \bar{X} = \frac{\mathbf{x}_1'\mathbf{y}_0}{\mathbf{x}_0'\mathbf{y}_0} = \frac{\sum x_1 y_0}{\sum x_0 y_0} \text{ and } \qquad (2a) \quad X_1 = \frac{\mathbf{x}_1'\mathbf{y}_1}{\mathbf{x}_0'\mathbf{y}_1} = \frac{\sum x_1 y_1}{\sum x_0 y_1} \text{ and likewise}$$

$$(3) \qquad Y_0 = \bar{Y} = \frac{\sum y_1 x_0}{\sum y_0 x_0} = \frac{\mathbf{x}_0'\mathbf{y}_1}{\mathbf{x}_0'\mathbf{y}_0} \text{ and } \qquad (3a) \quad Y_1 = \frac{\mathbf{x}_1'\mathbf{y}_1}{\mathbf{x}_1'\mathbf{y}_0} = \frac{\sum x_1 y_1}{\sum x_1 y_0}$$

the theorem states that the ratio of the two indices is given by

$$(4) \qquad \frac{X_1}{X_0} = \frac{Y_1}{Y_0} = \frac{\mathbf{x}_1'\mathbf{y}_1}{\mathbf{x}_0'\mathbf{y}_1} \cdot \frac{\mathbf{x}_0'\mathbf{y}_0}{\mathbf{x}_1'\mathbf{y}_0} = 1 + \frac{c_{xy}}{\bar{X} \cdot \bar{Y}} = 1 + \bar{c}_{xy} \text{ with the weighted covariance}$$

$$(5) \qquad c_{xy} = \sum \left(\frac{x_1}{x_0} - \bar{X}\right)\left(\frac{y_1}{y_0} - \bar{Y}\right) w_0 = \frac{\sum x_1 y_1}{\sum x_0 y_0} - \bar{X} \cdot \bar{Y}$$

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[1] A negative covariance ($P^P < P^L$) may arise from rational substitution among goods in response to price changes on a given (negatively sloped) demand curve. The less frequent case of a positive covariance is supposed to take place when the demand curve is shifting away from the origin (due to an increase of income for example).

[2] This is a *generalized* theorem of Bortkiewicz for the ratio X_1/X_0 of two linear indices. See von der Lippe (2007), pp. 194 – 196. The best known special case of this theorem is $X_0 = P^L$ and $X_1 = P^P$.

Using $X_1/X_0 = Y_1/Y_0$, the covariance c_{xy} can also be written as

(5a) $$c_{xy} = \overline{Y}(X_1 - \overline{X}) = \overline{X}(Y_1 - \overline{Y})$$

where $\overline{Y} = \sum \frac{y_1}{y_0} w_0$ and $\overline{X} = \sum \frac{x_1}{x_0} w_0$ and weights w_0 defined as $w_0 = x_0 y_0 / \sum x_0 y_0$.

Note that $\gamma_{xy} = \mathbf{x}_1'\mathbf{y}_1 / \mathbf{x}_0'\mathbf{y}_0 = X_0 Y_1 = Y_0 X_1$ is the product-moment $\mathbf{x}_1'\mathbf{y}_1 / \mathbf{x}_0'\mathbf{y}_0$ *around the origin* (0, 0) while $c_{xy} = \gamma_{xy} - \overline{X} \cdot \overline{Y} = X_0(Y_1 - Y_0) = Y_0(X_1 - X_0)$ is the *central* product-moment (around $\overline{X}, \overline{Y}$) or covariance and $\bar{c}_{xy} = \frac{\gamma_{xy}}{\overline{X} \cdot \overline{Y}} = \frac{Y_1}{Y_0} - 1 = \frac{X_1}{X_0} - 1$ may be called *centred* covariance. It can easily be seen that setting vectors $\mathbf{x}_0 = \mathbf{y}_0 = \mathbf{1} = [1 \ldots 1]$ eq. (5) reduces to the unweighted covariance $\frac{1}{n}\sum x_1 y_1 - \bar{x}_1 \bar{y}_1$.

The relationship between P_{01}^L and P_{01}^P now emerges when the assumptions concerning price and quantity vectors of row 1 in Table 1 are made

3. Some applications of the theorem

Note that the theorem allows for different representations of the difference of the same two formulas in terms of covariances. $X_1/X_0 = P_{01}^P / P_{01}^L$ or $Y_1/Y_0 = P_{01}^P / P_{01}^L$ can also be expressed as shown in row 2 and 3 of tab. 2. The vectors (p_1/p_0) and ($p_0q_0/\mathbf{\Sigma}$) are defined as follows

$$(p_1/p_0) = \begin{bmatrix} \frac{p_{11}}{p_{10}} & \cdots & \frac{p_{n1}}{p_{n0}} \end{bmatrix}, \text{ and } (p_0 q_0/\Sigma) = \begin{bmatrix} \frac{p_{10}q_{10}}{\sum p_{i0}q_{i0}} & \cdots & \frac{p_{n0}q_{n0}}{\sum p_{i0}q_{i0}} \end{bmatrix},$$

and ($p_0q_1/\mathbf{\Sigma}$) etc. correspondingly. The weights w_0 in the case of row 2a as well as row 2b are the same as in eq. 1. In fact the covariance in row 2a is simply the covariance c_{xy} of eq. (1) divided by Q_{01}^L, viz.

$$\frac{c_{xy}}{Q_{01}^L} = \sum_i \left(\frac{p_{i1}}{p_{i0}} - P_{01}^L \right) \left(\frac{q_{i1}}{q_{i0}} \frac{1}{Q_{01}^L} - 1 \right) \frac{p_{i0}q_{i0}}{\sum p_{i0}q_{i0}} = P_{01}^P - P_{01}^L,$$

which is completely in line with (5a) since $\overline{Y} = 1$ and $X_1 = P_{01}^P$, and $X_0 = \overline{X} = P_{01}^L$. Row 2b simply amounts to $\sum_i \left(\frac{q_{i1}}{q_{i0}} \frac{1}{Q_{01}^L} - 1 \right) \left(\frac{p_{i1}}{p_{i0}} - P_{01}^L \right) \frac{p_{i0}q_{i0}}{\sum p_{i0}q_{i0}}$.

Sometimes two index functions are related via a ratio with covariances in both, numerator and denominator. For example to demonstrate how Drobisch's unit value index

(6) $$P_{01}^{DR} = \frac{\sum p_1 q_1 / \sum q_1}{\sum p_0 q_0 / \sum q_0} = \frac{\tilde{p}_1}{\tilde{p}_0} = \frac{\mathbf{p}_1'\mathbf{q}_1}{\mathbf{p}_0'\mathbf{q}_0} : \frac{\mathbf{1}'\mathbf{q}_1}{\mathbf{1}'\mathbf{q}_0} = V_{01}/Q_{01}^D$$

where Q_{01}^D is the Dutot quantity index, is related to Dutot's price index

(6a) $$P_{01}^D = \frac{\sum p_1}{\sum p_0} = \frac{\bar{p}_1}{\bar{p}_0} = \frac{\mathbf{1}'\mathbf{p}_1}{\mathbf{1}'\mathbf{p}_0}$$

we need two steps (see row 3a and 3b) in order to arrive at (7).

Table 1

	Assumptions				Consequences			
	$\mathbf{x}_0$	$\mathbf{y}_0$	$\mathbf{x}_1$	$\mathbf{y}_1$	$X_0=\overline{X}$	X_1	$Y_0=\overline{Y}$	Y_1
1	$\mathbf{p}_0$	$\mathbf{q}_0$	$\mathbf{p}_1$	$\mathbf{q}_1$	P_{01}^{L}	P_{01}^{P}	Q_{01}^{L}	Q_{01}^{P}
2a	**1**	$(p_0q_0/\boldsymbol{\Sigma})$	(p_1/p_0)	$(p_0q_1/\boldsymbol{\Sigma})$	P_{01}^{L}	P_{01}^{P}	1	P_{01}^{P}/P_{01}^{L}
2b	$(p_0q_0/\boldsymbol{\Sigma})$	**1**	$(p_0q_1/\boldsymbol{\Sigma})$	(p_1/p_0)	1	P_{01}^{P}/P_{01}^{L}	P_{01}^{L}	P_{01}^{P}
3a	**1**	(1/n)	$\mathbf{p}_1$	$(q_1/\boldsymbol{\Sigma})$	$\overline{p}_1$	$\widetilde{p}_1$	1	$\widetilde{p}_1/\overline{p}_1$
3b	**1**	(1/n)	$\mathbf{p}_0$	$(q_0/\boldsymbol{\Sigma})$	$\overline{p}_0$	$\widetilde{p}_0$	1	$\widetilde{p}_0/\overline{p}_0$
4	$\mathbf{q}_0$	**1**	$\mathbf{q}_1$	$\mathbf{p}_1$	Q_{01}^{D}	Q_{01}^{P}	$\ddot{p}_1$ a)	$\widetilde{p}_1$
5	$\mathbf{p}_0$	$\mathbf{q}_0$	$\mathbf{p}_1$	**1**	P_{01}^{L}	P_{01}^{D}	$\sum p_0/\sum p_0q_0$ b)	$\sum p_1/\sum p_1q_0$
6	$\mathbf{q}_0$	$\mathbf{p}_0$	$\mathbf{q}_1$	$\breve{\mathbf{p}}_{01}$	Q_{01}^{L}	Q_{01}^{LE}	c)	c)

a) *row 4:* $\ddot{p}_1 = \sum p_{i1}\left(q_{i0}/\sum q_{i0}\right)$, note that $\widetilde{p}_1/\ddot{p}_1 = Q_{01}^{P}/Q_{01}^{D}$ (substituting $\mathbf{y}_1 = \mathbf{p}_1$ by $\mathbf{y}_1 = \mathbf{p}_0$ we get the corresponding relation between P^{DR} an P^{P} [instead of P^{DR} and P^{L}])

b) *row 5:* weights are here $w_0 = p_0q_0/\Sigma p_0q_0$, and it is clear that the weighted mean of the $1/q_0$ terms is $\overline{Y} = \sum p_0/\sum p_0q_0$.

c) *row 6:* $\overline{Y} = Y_0$ here is given as the price index $\overline{Y} = \sum \breve{p}_{01}q_0/\sum p_0q_0$ and Y_1 by the price index $Y_1 = \sum \breve{p}_{01}q_1/\sum p_0q_1$ index

The covariance in row 3a is given by $c_{xy}^{3a} = \sum_i (p_{i1}-\overline{p}_1)\left(\frac{q_{i1}}{\sum q_{i1}}-1\right)\frac{1}{n} = \widetilde{p}_1 - \overline{p}_1$ and quite similarly in row 3b we have $c_{xy}^{3b} = \sum_i (p_{i0}-\overline{p}_0)\left(\frac{q_{i0}}{\sum q_{i0}}-1\right)\frac{1}{n} = \widetilde{p}_0 - \overline{p}_0$, so that

(7) $$\frac{P_{01}^{DR}}{P_{01}^{D}} = \frac{\widetilde{p}_1/\widetilde{p}_0}{\overline{p}_1/\overline{p}_0} = \frac{\widetilde{p}_1/\overline{p}_1}{\widetilde{p}_0/\overline{p}_0} = \frac{1+\overline{c}_{xy}^{3a}}{1+\overline{c}_{xy}^{3b}}.$$

Using (6) and

(8) $$P_{01}^{L} = V_{01}/Q_{01}^{P}$$

allows to compare P_{01}^{DR} to P_{01}^{L} via comparing Q_{01}^{D} to Q_{01}^{P}, so that

$$\frac{P_{01}^{DR}}{P_{01}^{L}} = \frac{V_{01}}{Q_{01}^{D}} : \frac{V_{01}}{Q_{01}^{P}} = \frac{Q_{01}^{P}}{Q_{01}^{D}} = \frac{X_1}{X_0}$$ (see above row 4 in table 1).

In a similar vein we may compare P_{01}^{DR} to $P_{01}^{P} = V_{01}/Q_{01}^{L}$ via Q_{01}^{D} to Q_{01}^{L}.

In row 5 the Laspeyres index P_{01}^{L} is compared to the Drobisch index P_{01}^{D}. The relevant covariance now is between p_1/p_0 and $1/q_0$ with weights $w_0 = p_0q_0/\Sigma p_0q_0$. In CSW 1980, p. 18 we find a more complicated expression using two *unweighted* ($w_0 = 1/n$) covariances between p_1 and q_0 on the one hand (numerator) and p_0 and q_0 on the other hand (denominator) around means $\overline{p}_1$ and $\overline{q}_0$ (or $\overline{p}_0$ and $\overline{q}_0$ respectively). This shows that sometimes a choice can be made between more or less succinct and elegant formulas describing virtually the same relationship between two index functions.

In v. d. Lippe (2010) we could show that in principle all formulas comparing indices on the basis of covariances as developed in Diewert and v. d. Lippe (2010) can be viewed as special cases of the generalized Bortkiewicz theorem.

Finally it can be seen how the somewhat awkward historical index of the German economist Julius Lehr (1845 - 1894) for two adjacent periods 0, and 1 (Lehr is often seen as the fist who explicitly advocated chain indices) defined as

(9) $$P_{01}^{LE} = \frac{\sum p_{i1}q_{i1}}{\sum p_{i0}q_{i0}} \cdot \frac{\sum q_{i0}\breve{p}_{i,01}}{\sum q_{i1}\breve{p}_{i,01}} = \frac{V_{01}}{Q_{01}^{LE}}$$

is related to other linear indices. In (9) $\breve{p}_{i,12} = \frac{p_{i0}q_{i0} + p_{i1}q_{i1}}{q_{i0} + q_{i1}}$ so that $\frac{\breve{p}_{01}}{p_0} = \frac{q_0}{q_0 + q_1} + \frac{q_1}{q_0 + q_1}\frac{p_1}{p_0}$ is a linear transformation of price relatives p_1/p_0.

P_{01}^{LE} now can easily be related to $P_{01}^{L} = V_{01}/Q_{01}^{P}$ or to $P_{01}^{P} = V_{01}/Q_{01}^{L}$. In row 6 of table 1 we study the relationship $X_1/X_0 = P_{01}^{P}/P_{01}^{LE} = Q_{01}^{LE}/Q_{01}^{L}$. Thus the relation between these two indices depends on the covariance $\sum\left(\frac{q_1}{q_0} - \overline{X}\right)\left(\frac{\breve{p}_{01}}{p_0} - \overline{Y}\right)\frac{p_0q_0}{\sum p_0q_0}$. Upon setting $\mathbf{y}_0 = \mathbf{p}_1$ instead of $\mathbf{y}_0 = \mathbf{p}_0$ we get the relationship $X_1/X_0 = P_{01}^{L}/P_{01}^{LE} = Q_{01}^{LE}/Q_{01}^{P}$. As stated at the outset, it is generally assumed that a *negative* correlation between price relatives p_1/p_0 and quantity relatives q_1/q_0 is more likely than an positive correlation, so we can conclude that the same is true for a linear transformation of the p_1/p_0 (with a positive slope $q_1/(q_0 + q_1)$) and the q_1/q_0, and from this it follows that as (generally) $P_{01}^{P} < P_{01}^{L}$. so we may also expect $P_{01}^{P} < P_{01}^{LE} < P_{01}^{L}$.

The paper clearly shows that we may benefit a lot from the generalized theorem of Ladislaus von Bortkiewicz which describes the relationship between any two linear index functions in terms of (weighted) covariances between certain variables such as prices or quantities or price and quantity relatives.

References

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