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Divisia Index

Let P(τ) denote the continuous price (index) function varying continuously over time and let the quality (index) function Q(τ) be defined analoguously. In the same way we may assume that there exists a value (index) function V (τ) which is for each moment in time an aggregate of n commodities (i=1,2,...,n)

(6.1) $$V(\tau) = \sum_{i=1}^{n} p_i(\tau) q_i(\tau)$$

and for which the factor-reversal test holds

(6.2) $$V(\tau) = P(\tau)\ Q(\tau)$$

Note that the factor reversal property is <u>not</u> an implication of time (t) being a continuous variable. It is merely a consequence of the definition of P(τ). We may easily think of a definition of P(τ) for which eq. 6.2 would not hold, as will be shown later.

By eq. 6.2 the indices P(τ) and Q(τ) are defined only implicitly, no instruction is given, how to calculate the indices. Such an instruction can only be derived by the following considerations.

Differential changes of V(τ) are

$$dV(\tau) = \sum q_i(\tau)\ dp_i(\tau) + \sum p_i(\tau)\ dq_i(\tau)$$

Dividing by V(t) as given by eq. 6.1 gives the growth rate (logartihmic deviviate) of V(τ)

$$\frac{dV}{V} = \frac{\sum q\ dp}{\sum qp} + \frac{\sum p\ dq}{\sum pq}$$, omitting τ and i for convenience.

The factor reversal condition is imposed ever since we identify the first (second) term as growth rate of the price (volume) index.

$$dV = Q\ dP + P\ dQ$$

and therefore

(6.2a) $$\frac{dV}{V} = \frac{dP}{P} + \frac{dQ}{Q}$$ we may define

(6.3a) $$\frac{dP(\tau)}{P(\tau)} = \frac{d\ \ln\ P(\tau)}{d\tau} = \frac{\sum q\ dp}{\sum qp}$$ and

(6.3b) $$\frac{dQ}{Q} = d\ \ln\ Q = \frac{\sum p\ dq}{\sum pq}$$

The Divisia price index P(t) and the Divisia quantity index Q(t) is given by solving these differential equations. The name integral index stems from the fact that P(t) is by definition

(6.4) $$P(t) = \int_0^t \frac{\sum q_i(\tau)\ dp_i(\tau)}{\sum q_i(\tau) p_i(\tau)} d\tau$$

and Q(t) respectively. The pair of Divisia indices satisfies the factor reversal test by definition. It is this criterion that allows to seperate the two differentials (in price and in quantity) and to identify them as price and quantity index respectively.

In the integration of $\frac{dP(\tau)}{P(\tau)}$ the solution of which gives the price index P(t) quantity is assumed to be constant and when $\frac{dQ(\tau)}{Q(\tau)}$ is integrated price is assumend to be constant. This idea may give us the chance to outline a relationship between this approach and some well known formulas before continuing the general discussion of Divisia's approach.

The integration of the price differential given constant or proportional quantities $q(\tau) = \lambda q(0)$ (the subscript i denoting the commodity will be deleted for convenience)

$$\frac{dP(\tau)}{P(\tau)} = \frac{\sum q(\tau)dp(\tau)}{\sum q(\tau)p(\tau)} = \frac{\sum \lambda q(0)dp(\tau)}{\sum \lambda q(0)p(\tau)} = \frac{\sum q(0)dp(\tau)}{\sum q(0)p(\tau)} = \frac{d\sum q(0)p(\tau)}{\sum q(0)p(\tau)}$$

Integrating $\int_0^t \frac{dP(\tau)}{P(\tau)} d\tau$ we get $\log P(t) = \log [\sum q(0)p(t)] + c$

with an arbitrary constant c. Assuming P(0) the price level with prices p(0) in the base period we get

$$\log \frac{P(t)}{P(0)} = \log P(t) - \log P(0) = \log \frac{\sum p(t)q(0)}{\sum p(0)q(0)}$$

and finally $\frac{P(t)}{P(0)} = p_{0t} = \frac{\sum p(t)q(0)}{\sum p(0)q(0)}$ which is the familiar Laspeyres' price index.

In a similar way Paasche's formula can be deduced by setting $q(\tau)$ equal to $\lambda q(t)$. It is also possible to derive Fisher's ideal index[1] from the differential equation with assumptions, however, which may not have such an obvious interpretation.

Substituting forward differences $\Delta P_t = P_{t+1} - P_t$ instead of the differential dp in eq. 6.3a we get a practical approximation applicable to discrete time intervals

$$\frac{\Delta P_t}{P_t} = \frac{\sum q_t \Delta p_t}{\sum q_t p_t} \qquad \frac{P_{t+1}}{P_t} - 1 = \frac{\sum q_t p_{t+1}}{\sum q_t p_t} - \frac{\sum q_t p_t}{\sum q_t p_t}$$ and finally

(6.5) $$\frac{P_{t+1}}{P_t} = P^{LC}_{t+1} = \frac{\sum p_{t+1} q_t}{\sum p_t q_t}$$

which is the Laspeyres-type chain price index. In a similar manner we may easily derive the Laspeyres quantity chain index Q^{LC}_t from eq. 6.3b and the two versions of Paasche chain indices by using backward differences $\Delta^* P_t = P_t - P_{t-1}$ and respectively[2].

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[1] See K.B. Banerjee, Cost of Living Index Numberes, p. 127 f.

[2] See R.G.D. Allen, op.cit., p. 181

$$\frac{\Delta^* P_t}{P_t} = 1 - \frac{P_{t-1}}{P_t} = 1 - \left(\frac{P_t}{P_{t-1}}\right)^{-1} = \frac{\sum q_t \Delta^* p_t}{\sum q_t p_t} = 1 - \frac{\sum q_t p_{t-1}}{\sum q_t p_t} = 1 - \left(P_t^{PC}\right)^{-1}$$

and Q_t^{PC} can be interpreted correspondingly.

Thus we may regard chain indices as being a practical (discrete time) approximations of Divisia indices (DI). In many texts[3] the conclusion drawn from this finding is: since chain indices are discrete approximations of DI they also have the same properties, that is, they satisfy the factor reversal and the chain test, as DI does.

Due to eq. 6.2a DI satisfies the factor reversal test (eq. 6.2) but applying the same substitution (of Δp_t and Δq_t for dp and dq in eq. 6.2a) we get

$$\frac{\sum q_t \Delta p_t}{\sum q_t p_t} + \frac{\sum p_t \Delta q_t}{\sum p_t q_t} = \frac{1}{\sum p_t q_t}\left(\sum p_{t+1} q_t - \sum p_t q_t + \sum p_t q_{t+1} - \sum p_t q_t\right)$$

which is unequal

$$\frac{\Delta v_t}{v_t} = \frac{v_{t+1} - v_t}{v_t} = \frac{v_{t+1}}{v_t} - 1 = \frac{\sum p_{t+1} q_{t+1}}{\sum p_t q_t} - 1$$

Therefore a pair of Laspeyres chain index numbers will not meet the factor reversal test (eq. 6.2)

$$\frac{\sum p_{t+1} q_t}{\sum p_t q_t} * \frac{\sum q_{t+1} p_t}{\sum p_t q_t} \neq \frac{\sum p_{t+1} q_{t+1}}{\sum p_t q_t}$$

A pair of Paasche chain indices will not fulfill eq. 6.2 either.

Taking P^{LC} and Q^{PC} (or P^{PC} and Q^{LC}) we get, however,

$$P_{t+1}^{LC} Q_{t+1}^{PC} = \frac{\sum p_{t+1} q_t}{\sum p_t q_t} * \frac{\sum q_{t+1} p_{t+1}}{\sum q_t p_{t+1}} = \frac{\sum p_{t+1} q_{t+1}}{\sum p_t q_t} = \frac{V_{t+1}}{V_t} \quad \text{and}$$

$$P_{t+1}^{PC} Q_{t+1}^{LC} = \frac{\sum p_{t+1} q_{t+1}}{\sum p_t q_{t+1}} * \frac{\sum q_{t+1} p_t}{\sum q_t p_t} = \frac{V_{t+1}}{V_t} \quad .$$

Thus the relationship between the Laspeyres and the Paasche form, as one being the "antithesis" of the other, introduced already in the binary case of "direct" indices, does also hold with respect a link of chain indices.

The fact that the chain index approach may be regarded as being a discrete time approximation of the continuous time Divisia approach should not be misunderstood in the sense that the favourable properties of DI will also be transfered to the chain index. By substituting differences for differentials we can not conclude that this operation will not do any harm and destroy some fundamental relationships that apply to the continuous time approach.

[3] This is done at least implicitly in the quoted textbooks of Allen (p. 181 f) and Banerjee (p.