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Notes on GEKS and RGEKS indices

First introduced by Ivancic, Fox and Diewert (Ottawa Group Meeting in Neuchâtel 2009). Adopts GEKS¹-method (for international comparisons) to create transitive indices by averaging direct (not chained) Fisher indices for the purpose of intertemporal comparisons.

1. Transitivity

This means that *all* indirect comparisons between A and B (via C, D etc.) are consistent with *the* [unique] direct one, viz. P_{AB} . A most restrictive property which appears justified in the international framework, but it is "over-ambitious" in the intertemporal situation where only *some*, not *all* indirect comparisons are relevant, viz. those between adjacent intervals as they are used in the chain index method (i.e. over a sequence, like 0-1-2-3-4... rather than indirect 0-5-3-8-2 ...). It seems reasonable to compare 2013 to 2010 indirectly only via 2011 and 2012,² while via 1868 and 2018 will be quite pointless.

In this respect the (international) analogue of chain indices (for periods in time) is the Minimum Spanning Tree (MST) method in that it defines *one* particular sequence of binary comparisons (defined by similarity of countries and chronological order respectively) and does not require consistency of all sequences of binary comparisons. Moreover chain indices can be viewed as limiting case of rolling GEKS (or RGEKS) indices.

2. Data requirements (GEKS index is the most demanding)

Three types of index functions in the order of increasing complexity and data requirements, chain indices \bar{P}_{0t} , direct indices P_{0t} , and GEKS indices.

¹ Gini, Eltető, Köves, Szulc

² Also the RGEKS method proceeds this way following the course of time

GEKS-index-formula requires a time reversible index numbers, like P_{0t}^F (Henceforth all indices are Fisher index functions and simply denoted by P_{0t}) as building blocks, and is the most difficult to compile.

To calculate	requires
1. chain Fisher C_{06}	successively only P_{01} , then P_{12} , then P_{23} , ..., P_{56} , thus prices and quantities of only two adjacent periods at a time
2. direct Fisher P_{06}	prices and quantities of two (possibly distant) periods (here: 0 and 6); to ensure identity of goods over a long interval in time may prove problematic
3. GEKS G_{06}	in addition to P_{06} also P_{01} , P_{16} , P_{02} , P_{26} , P_{03} , P_{36} , P_{04} , P_{46} , P_{05} , P_{56}

The GEKS formulas become ever more complicated the longer the time series is from which they are calculated (the greater m is). Chain indices are the easiest to compile, however, inferior to direct indices as they are path dependent and violate "pure price comparison" (reflecting solely price movement).

$G_{02(m=6)}$ for example is built with $2m - 3 = 9$ Fisher indices as building blocks which in turn are made of 18 indices so that we have to compile 18 ratios of sums of products.

It also should be borne in mind that most of what is conceived as index theory (for example in the ways of utility maximization on a given preference function etc.) is aimed at a *direct* index P_{0t} comparing 0 and t , not at the chain or GEKS index.

3. Why are GEKS indices transitive

A (standard) GEKS index, say $G_{02(m=3)}$ can be written as ratio of two levels π_2/π_0 where $\pi_2 = (P_{02}P_{12}P_{22})^{1/3}$ and $\pi_0 = (P_{00}P_{10}P_{20})^{1/3}$, however, this no longer applies to RGEKS indices. GEKS indices are only transitive for a given common m . For example $G_{02(m=4)} \neq G_{01(m=3)} G_{12(m=3)}$.

4. GEKS index not uniquely determined; no "target index" against which to evaluate an index

The result for G_{03} from a series going from 0 to 3 (so that $m = 4$ periods are involved), that is $G_{03(m=4)}$ for example, will in general differ from the GEKS index for the same two periods, 0 and 3 when it is calculated from a series going to $t = 4$ or $t = 5$ etc. with consequently $m = 5$ or $m = 6$ etc. periods involved, so that $G_{03(m=4)}$ and $G_{03(m=5)}$ and $G_{03(m=6)}$ will in general yield different results for the same comparison of 3 to 0.

All these indices are equally legitimate. The GEKS method fails to provide a unique "drift-free" or "target" series of index numbers, unless m is fixed. However, no theory of *the* correct m exists.

5. Updating ($G_{0t} \rightarrow G_{0,t+1}$) standard GEKS indices vs. updating of chain indices

To proceed from $G_{02(m=3)} = ((P_{02})^2 P_{01} P_{12})^{1/3}$ to $G_{03(m=4)} = ((P_{03})^2 P_{01} P_{13} P_{02} P_{23})^{1/4}$ we need three more indices, viz. P_{03} , P_{13} , and P_{23} . In the case of a chain index we only need P_{23} . To move from $G_{03(m=4)}$ to $G_{04(m=5)}$ requires four new indices. To update the china index $C_{03} \rightarrow C_{04}$ again requires one index only (P_{34}).

6. Updating of GEKS indices requires re-computing of formerly computed indices

This is not necessary in the case of chain indices. For example $G_{02(m=4)}$ differs from $G_{02(m=3)}$ by $P_{03}P_{32}$. In general: given that t is the additional $(m+1)^{\text{th}}$ period we simply have to multiply by $P_{0t}P_{tk}$ and take the $(m+1)^{\text{th}}$ root. This implies $P_{0t}P_{tk} = (G_{0k(m=t+1)})^{1/t+1} / (G_{0k(m=t)})^{1/t}$.

7. Why and how "rolling"?

In order to overcome such difficulties and work uniformly with a fixed m it became common to combine the GEKS method with a "rolling" device so that the calculation is in the first ($w = 1$) window based on periods 0 to period $m-1$, then (window $w = 2$) from 1 to m , then in $w = 3$ from 2 to $m+1$ etc. Assume $m = 3$ where the

first window covers periods 0, 1, and 2. Then $G_{03(w=1, m=3)}$ is the first index which requires an estimate using a link $L_{23} = G_{13(w=2, m=3)}/G_{12(w=2, m=3)}$ so that the estimate is $G_{03(w=1, m=3)}^* = G_{02(w=1, m=3)}L_{23}$.

	window w = 1	window w = 2	window w = 3
0	$G_{00(w=1, m=3)} = P_{00} = 1$		
1	$G_{01(w=1, m=3)}$	$G_{11(w=2, m=3)} = P_{11} = 1$	
2	$G_{02(w=1, m=3)}$	$G_{12(w=2, m=3)}$	$G_{22(w=3, m=3)} = P_{22} = 1$
3	$G_{03}^* = G_{02}L_{23}$	$G_{13(w=2, m=3)}$	$G_{23(w=3, m=3)}$
4	$G_{04}^* = G_{02}L_{23}L_{34}$		$G_{24(w=3, m=3)}$

The scheme demonstrates that

1. Successive windows have an overlap of $m-1$ (here $m-1=2$) periods so that a number of (that is $m-1$) links may be formed for a transition between the same two periods (an alternative to L_{23} would be $L_{23}^* = G_{23(w=3, m=3)}/G_{22(w=3, m=3)} = G_{23(w=3, m=3)}$).
2. G_{04}^* requires two links, again $L_{34} = G_{24(w=3, m=3)}/G_{23(w=3, m=3)}$ will not in general equal $L_{34}^* = G_{34(w=4, m=3)}/G_{33(w=4, m=3)} = G_{34(w=4, m=3)}$.
3. Due to the fact that $m=3$ windows provide $m-1=2$ links for the same transition $2 \rightarrow 3$ or $3 \rightarrow 4$ etc. GEKS indices are no longer independent of the base, unlike chain indices which satisfy $C_{0s}/C_{0r} = C_{1s}/C_{1r} = C_{2s}/C_{2r} = \dots$ by construction. It can also easily be seen that
4. a chain index is the limiting case of $m=2$ of a rolling GEKS index.

Not surprisingly RGEKS and chain indices have some properties in common: path dependence (i.e. no transitivity) and lack of proportionality (and thereby identity). Numerical examples show that a G_{0t}^* index may violate identity (all prices $p_{i0} = p_{it}$) as soon as two or more links $L_{s,s+1}$ are involved in computing G_{0t}^* .

9. More ambiguities and intransitivity

a) As m increases we get more ambiguities of the sort referred to in sec. 8. For example for G_{34} we can form two estimates based on $m = 3$ windows (windows 3 and 4 covering periods 2 to 4 and 3 to 5 respectively, so that both windows cover the two periods, 3 and 4), three estimates with $m = 4$ windows and four with $m = 5$ windows, altogether 10 estimates. Note that by contrast to sec. 8 we refer here to a single index, viz. G_{34} , not to a series $G_{01}, G_{02}, \dots, G_{0,m-1}, G_{0m}^*, G_{0,m+1}^*, \dots$

b) As soon as a linking is involved the RGEKS index will differ from the corresponding standard GEKS index which is transitive by definition. Hence the RGEKS index is no longer transitive. It can easily be seen with the small $m = 3$ that $G_{03(m=3)}^* \neq G_{03(m=4)}$. In particular $G_{03(m=3)}^*$ cannot be written as a ratio of two "levels" that is $\pi_3/\pi_0 = (P_{03}P_{13}P_{23})^{1/3}/(P_{00}P_{10}P_{20})^{1/3}$. Also $G_{04(m=4)}^* \neq (P_{04}P_{14}P_{24}P_{34})^{1/4}/(P_{00}P_{10}P_{20}P_{30})^{1/4}$. and of course $G_{04(m=4)}^* \neq G_{04(m=5)}$.

c) Violation of identity can easily be seen. When for all $i = 1, \dots, n$ commodities holds $p_{i0} = p_{i4}$ and $q_{i0} = q_{i4}$ (and therefore $P_{04} = 1$) then $G_{04(m=3)}^* = G_{02(m=3)}L_{23}L_{34} = ((P_{02}P_{34})^2P_{01}P_{13}P_{23}P_{24})^{1/3}$ reduces to $((P_{30})^2P_{01}P_{02}P_{13}P_{23})^{1/3}$ which may well differ from unity.

10. Cycles and trends

It can be shown by means of a small numerical example that when prices show a cyclical movement of k periods but no trend, RGEKS indices ($m \neq \lambda k, \lambda = 1, 2, \dots$) may well (just like chain indices) fluctuate around a positively or negatively sloped trend (although the underlying price data don't show a trend).

The example was

t = 0		t = 1		t = 2		t = 3		t = 4	
p	q	p	q	p	q	p	q	p	q
2	10	4	12	3	20	1	16	2	10
5	20	3	15	4	10	4	12	5	20

A negatively sloped trend can easily be seen in the following figure

