Prof. Dr. Peter von der Lippe

Structure of the MEDSTAT Course and of the book "Index Theory and Price Statistics"

Figure 1.1.1 a

Index-theory and (practice of official) price statistics

Theory of (price) index numbers
- General introduction and elementary price index theory chap. 1

(Official) price statistics
- Price collection and index systems in official statistics, quality adjustment etc. chap. 4

- Intertemporal comparisons
- International comparisons chap. 8*
- Deflation and aggregation chap. 5
- Price level measurement

- Direct binary comparisons
- Chain index approach chap. 7
- Consumer prices (CPI) sec 6.1 + 6.2
- PPI and unit value indices sec 6.3 – 6.4

Approaches to index theory chap. 2**
- Axioms and more index formulas chap. 3

* not presented in this course
** e.g. formal index theory (focusing on mathematical properties of index functions) and economic theory of index numbers (aiming at a microeconomic foundation of index formulas).

Figure 1.1.1 b: Structure of formal index theory

Formal index theory: Intertemporal comparisons*

- Direct binary comparisons
- Chain approach; sec. 2.5, ch. 7
- Price level measurement
- Deflation of aggregates

Next page: structure of the book (without section 6.5: Employment cost index)
# von der Lippe: Handouts for the Medstat-Course Price Indices and Price Statistics

## 1. General introduction and elementary price index theory

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<th>Topic</th>
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<td>1.3</td>
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## 2. Approaches to index theory

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<td>2.3</td>
<td>The stochastic approach in price index theory</td>
</tr>
<tr>
<td>2.4</td>
<td>Economic approach (the &quot;true cost of living index&quot;, COLI)</td>
</tr>
<tr>
<td>2.5</td>
<td>Chain indices and Divisia's approach (general introduction)</td>
</tr>
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<td>2.6</td>
<td>Additive models: Stuvel's and Banerjee's approach</td>
</tr>
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## 3. Axioms and more index formulas

<table>
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<th>Topic</th>
</tr>
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<tbody>
<tr>
<td>3.1</td>
<td>The axiomatic approach and some fundamental axioms</td>
</tr>
<tr>
<td>3.2</td>
<td>Fundamental axioms and their interpretation Monotonicity, additivity, linear homogeneity etc.</td>
</tr>
<tr>
<td>3.3</td>
<td>Systems of axioms (Fisher, Eichhorn and Voeller etc.)</td>
</tr>
<tr>
<td>3.4</td>
<td>Log-change indices I (Törnquist)</td>
</tr>
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<td>3.5</td>
<td>Log-change indices II (Vartia)</td>
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<td>3.6</td>
<td>Ideal indices (factor reversibility) and Theil's &quot;best linear index&quot;</td>
</tr>
</tbody>
</table>

## 4. Price collection, quality adjustment and sampling in official statistics

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<thead>
<tr>
<th>Sub-section</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
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<td>4.1</td>
<td>The set up of a system of price quotations and price indices in official statistics</td>
</tr>
<tr>
<td>4.2</td>
<td>Quality adjustment in price statistics</td>
</tr>
<tr>
<td>4.3</td>
<td>Sampling in price statistics</td>
</tr>
</tbody>
</table>

## 5. Deflation and aggregation

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<tr>
<th>Sub-section</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>Introduction into deflation methods</td>
</tr>
<tr>
<td>5.2</td>
<td>Deflation in volume terms, aggregation and double deflation</td>
</tr>
<tr>
<td>5.3</td>
<td>Harmonization of deflation methodology in Europe</td>
</tr>
<tr>
<td>5.4</td>
<td>Fisher's &quot;ideal&quot; index as deflator</td>
</tr>
</tbody>
</table>

## 6. Consumer prices and quality adjustment

<table>
<thead>
<tr>
<th>Sub-section</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1</td>
<td>The Consumer Price Index (CPI) and the Harmonized Index of Consumer Prices (HICP)</td>
</tr>
<tr>
<td>6.2</td>
<td>Some controversial issues in inflation measurement (core inflation, asset inflation etc.)</td>
</tr>
<tr>
<td>6.3</td>
<td>Producer Price Indices (PPI)</td>
</tr>
<tr>
<td>6.4</td>
<td>Price indices and unit value indices, foreign trade and wage indices</td>
</tr>
</tbody>
</table>

## 7. Chain index approach

<table>
<thead>
<tr>
<th>Sub-section</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1</td>
<td>Chain indices: arguments pro and con</td>
</tr>
<tr>
<td>7.2</td>
<td>Properties of chain indices</td>
</tr>
</tbody>
</table>
Chapter 1 General introduction and elementary price index theory

1.1. Fundamental principles

a) The concept of a price index

Definition: A price index $P_{0t}$ is a function $P: \mathbb{R}^{kn} \rightarrow \mathbb{R}$ mapping $k = 4$ real valued vectors with $n$ dimensions

$$P_{0t} = P(p_0, q_0, p_t, q_t)$$

(1.1.1)

into a one dimensional positive real number for comparative purposes. The function $P( )$ should satisfy certain functional equations (= axioms) and (in order to) have a meaningful interpretation.

The word base is ambiguous because it may refer to the period to which
1. we compare the current state (reference base), or to which
2. the weights refer (weight base).

b) Objectives and methodological principles of price statistics, the concept of deflation

Deflation: Aggregate at current (period $t$) prices,

$$V_t = \sum p_i q_{it}$$

which is called a value, $V_t$ (or "nominal" aggregate). The same (with respect to the selection of commodities and their quantities) aggregate valued at constant prices of the base period 0

$$Q_t = \sum p_i q_{it}$$

(1.1.3)

is called a volume (or "real" aggregate)

$$V_{0t} = V_t / V_0$$

value index (value ratio).

Fundamental methodological principles of price statistics and price indexes (see fig. 1.1.2)

a) the selection of "representative" prices, and

b) the principle of "pure price comparison".

To better understand the two conflicting principles representativity (R) and pure price comparison (P) we should start defining "Price determining characteristics" (PDC). The PDCs are the quantity and quality of the commodity, the shop (outlet) in which the sale takes place,
a bonus granted or services rendered in connection with the sale if applicable, arrangements made concerning delivery, availability of spare parts, insurance etc.

As conditions inevitably change, obviously both principles are hard to reconcile. In practice always some compromise in one way or another between R and P has to be found which tries to comply with both conflicting principles acceptably. No solution can meet each of them to full satisfaction.

Note that this conflict also applies to index formulas: A reasonable compromise is - in our view - neither a chain index (focusing on R at the expense of P) nor keeping weights of a Laspeyres index unchanged for quite a long period in time (that is pursuing exclusively P and neglecting R unduly) but rather a Laspeyres index in which weights are reviewed and updated in intervals of say five years or so.

**Figure 1.1.2: Uses of price statistics and price indices**

<table>
<thead>
<tr>
<th>a) Purposes of price indices in intertemporal comparisons</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Analytical use:</strong> the measurement of &quot;price levels&quot; on certain markets</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>b) Methodological principles of price indices and price statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Representativeness</strong> (R)</td>
</tr>
<tr>
<td>Select contracts governed by the most common (frequent) and typical PDCs</td>
</tr>
</tbody>
</table>

*or: Representativity

c) Simple comparisons (a single commodity)

Three ways of describing the change in an individual price of commodity i:

1. **Price relatives** (≈ growth factors) with reference to a base period price \( p_{0t} \) (eq. 1.1.5) and links (eq. 1.1.6).

2. **Log-changes** (≈ growth rates referring to a logarithmic mean as the reference value) → eq. 1.1.8, → leading to "log-change indices" (sec. 3.4 and 3.5)

3. **Differentials** with respect to time \( dp_i(t)/dt \) terms → Divisia index

3a. and (less common) **Absolute differences** (= variations, Hillinger).

**Relatives (related to a fixed base)** as opposed to links (link relatives) = variable base

\[
(1.1.5) \quad a_{0t} = a_{0t} = \frac{p_t}{p_{0t}}.
\]

Quantity relative \( b_{0t} = \frac{q_t}{q_0} \) and a value relative \( c_{0t} = \frac{v_t}{v_0} = \frac{p_t q_t}{p_0 q_0} = a_{0t} b_{0t} \).

**Links (chain base) and Log changes**

\[
(1.1.6) \quad \ell_t = \frac{p_t}{p_{t-1}} = a_{t-1,t}, \quad (1.1.7) \quad a_{0t} = \ell_1 \ell_2 \cdots \ell_{t-1} \ell_t.
\]
The terms

$ Da_{0t} = \ln\left(\frac{p_t}{p_0}\right) $ or

$ D\ell_t = \ln\left(\frac{p_t}{p_{t-1}}\right) $.

are called log-changes. There exists an appropriate mean of terms like $ Da_{0t} $, that is the logarithmic mean of two positive numbers, $ x $ and $ y $ ($ x \neq y $)

$ L(x,y) = \frac{y-x}{\ln(y/x)} = L(y,x) \quad \text{logarithmic mean.} $ (1.1.9)

Log changes can be interpreted as growth rates.

There are some index formulas based on log changes and $ L $, like the Törnquist index ($ P_T $) or Vartia indices, but (with the exception of $ P_T $) their role in official statistics was rather small until now.

### Table 1.1.1: Axioms satisfied by price and quantity relatives (fixed base)

<table>
<thead>
<tr>
<th>no</th>
<th>axiom</th>
<th>definition and interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>identity</td>
<td>$ a_{00} = a_{tt} = 1 \quad \text{uniqueness of the reference point} $</td>
</tr>
<tr>
<td>2</td>
<td>dimensionality (price</td>
<td>$ \frac{\lambda p_t}{\lambda p_0} = \frac{p_t}{p_0} \quad a_{0t} \text{ is independent of the currency in which the }</td>
</tr>
<tr>
<td></td>
<td>dimensionality)</td>
<td>prices are expressed } b)</td>
</tr>
<tr>
<td>3</td>
<td>commensurability a)</td>
<td>$ \frac{\lambda p_t}{\lambda p_0} = \frac{p_t}{p_0} \quad \text{independence of the unit of quantity to which }</td>
</tr>
<tr>
<td></td>
<td></td>
<td>the price of commodity i refers} b)</td>
</tr>
<tr>
<td>4</td>
<td>time reversal test</td>
<td>$ a_{to} = \frac{1}{a_{0t}} \quad \text{consistency of relatives with different base periods}</td>
</tr>
<tr>
<td>5</td>
<td>factor reversal test</td>
<td>$ c_{0t} = a_{0t}b_{0t} \quad \text{the value change is decomposable in price change and quantity change}</td>
</tr>
<tr>
<td>6</td>
<td>transitivity</td>
<td>$ a_{0t} = a_{0s}a_{st} \quad \text{for all three periods 0, s and t} $</td>
</tr>
</tbody>
</table>

a) if only $ n = 1 $ commodity is involved the mathematical representation of 2 and 3 can not be distinguished.
b) in both periods, 0 and t.

d) Aggregative comparisons (two or more commodities) and unit values

$$ \bar{p}_t = \frac{\sum p_t q_{it}}{\sum q_{it}} = \frac{p_t}{\sum q_{it}} \quad \text{(unit value in t)} $$

Example 1.1.1

Imagine an economy with only two industries A and B, and wages of $10 and $16 paid at base period:

<table>
<thead>
<tr>
<th>situation in base period</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>industry</td>
<td>wage</td>
<td>hours</td>
<td>payment</td>
</tr>
<tr>
<td>A</td>
<td>10</td>
<td>50</td>
<td>500</td>
</tr>
<tr>
<td>B</td>
<td>16</td>
<td>50</td>
<td>800</td>
</tr>
<tr>
<td>sum*</td>
<td>13</td>
<td>100</td>
<td>1300</td>
</tr>
</tbody>
</table>

* or average

It would be highly misleading to compare simply the average wage per hour presently paid with the average wage formerly paid at the base year. Assume two alternative (presented for demonstrative purposes) situations in t
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<table>
<thead>
<tr>
<th>situation in t</th>
<th>alternative situation in t</th>
</tr>
</thead>
<tbody>
<tr>
<td>industry</td>
<td>wage</td>
</tr>
<tr>
<td>A</td>
<td>15</td>
</tr>
<tr>
<td>B</td>
<td>24</td>
</tr>
<tr>
<td>total</td>
<td>15.9</td>
</tr>
</tbody>
</table>

Structural change has to be eliminated by some method of weighting (for example with a constant base year structure, like in a Laspeyres type index). Indices differ from averages ("unit values") by their invariance to a structural change and thus their ability to provide "pure" comparisons.

1.2. Unweighted indices

<table>
<thead>
<tr>
<th>a) Dutot’s index and Drobisch’s unit value index</th>
<th>c) Commensurability and time reversal test</th>
</tr>
</thead>
<tbody>
<tr>
<td>b) Carli’s index formula</td>
<td>d) Stochastic and aggregative approach</td>
</tr>
</tbody>
</table>

a) Dutot’s index and unit value index of Drobisch

\[
P_{0t}^D = \frac{\bar{p}_t}{\bar{p}_0} = \frac{1}{n} \sum p_{it} = \frac{\sum p_{it}}{\sum p_{i0}} \quad \text{(price index of Dutot 1738).}
\]

\[
P_{0t}^{UD} = \frac{\bar{p}_t}{\bar{p}_0} = \frac{\sum p_{ai} q_{it}}{\sum p_{i0} q_{i0}} \quad \text{(unit value index of Drobisch 1871).}
\]

Both formulas have in common that they are ratios of absolute figures, that is of average prices expressed in €, $, £ or the like. Such indices violate commensurability (reason: The sums \( \sum p_{iq_t} \) and \( \sum p_{q_0} \) are not affected by a change in the physical units of quantities, but the sums \( \sum q_{it} \) and \( \sum q_{i0} \) are and so are the sums \( \sum p_{it} \) and \( \sum p_{i0} \)).

\( P_{UD}^D \) has three additional shortcomings (→ ex. 1.2.1):

- \( P_{UD} \) does not meet the mean value condition,
- the sums \( \sum q_{it} \) and \( \sum q_{i0} \) are in general not defined,
- \( P_{UD} \) can indicate a change of the price level although all prices remain unchanged, simply because quantity changed (in level or in structure). Hence \( P_{UD}^D \) violates identity.

Example 1.2.1

Consider the following prices and quantities of two commodities, A and B

<table>
<thead>
<tr>
<th>i</th>
<th>( p_{i0} )</th>
<th>( p_{it} )</th>
<th>( q_{i0} )</th>
<th>( q_{it} )</th>
<th>( q_{it}^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>10</td>
<td>15</td>
<td>5</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>30</td>
<td>35</td>
<td>2</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>( \Sigma )</td>
<td>40</td>
<td>50</td>
<td>7</td>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>

Index of Dutot: \( P_{0t}^D = \frac{\bar{p}_t}{\bar{p}_0} = \frac{25}{20} = \frac{\sum p_{i} / \sum p_{0}}{40 / 40} = 1.25 \). Now assume prices \( p_{A0} \) and \( p_{At} \) refer to quarts and price statistics changes to a quotation in terms of gallons (prices and quantities of B remain unchanged). With prices of A on the basis of gallons we get

\[ p_{D}^{t} = \frac{60 + 35}{40 + 30} = \frac{95}{70} = 1.357 \] indicating a rise of prices by more than 25%. The unit value index \( P^{UD} \) amounts to \( P_{0t}^{UD} = \frac{(120 + 140)/12}{(50 + 60)/7} = \frac{260/12}{110/7} = \frac{21.67}{15.71} = 1.3788 \).

Assume now prices remained unchanged, so that \( p_{At} = 10 \) and \( p_{Bt} = 30 \). A reasonable index should be unity, however \( P_{0t}^{UD} = \frac{200/12}{110/7} = \frac{16.67}{15.71} = 1.0606 \), indicating a rise by 6%. Since

\[ (1.2.2a) \quad P_{0t}^{UD} = \frac{\sum p_{it} q_{it}}{\sum q_{it}} \sum p_{i0} \frac{q_{i0}}{\sum q_{i0}} = \frac{\sum p_{it} \alpha_{it}}{\sum p_{i0} \alpha_{i0}} \]

where weights \( \alpha \) are reflecting the structure of quantities. As weights \( \alpha_{it} \) differ from \( \alpha_{i0} \) such that commodity B, which is cheaper than A, gets a weight \( \alpha_{it} > \alpha_{i0} \) the result is \( P^{UD} < 1 \). Taking quantities \( q^{*} \) we obtain: \( P_{0t}^{UD} = \frac{(100 + 60)/12}{110/7} = \frac{13.33}{15.71} = 0.8485 \).

Note: \( P_{it}^{UD} = \frac{\sum p_{i0} \alpha_{i0}}{\sum p_{i0} \alpha_{i0}} \); \( P_{0t}^{UD} = \frac{\sum p_{i0} \alpha_{i0}}{\sum p_{i0} \alpha_{i0}} \)

An analysis of \( P^{UD} \) also shows why the ratio of "average wages" (virtually unit values with quantities \( q \) being numbers of employees) in ex. 1.1.1 is unacceptable. Since in this example \( \sum q_{t} = \sum q_{0} = 100 P^{UD} \) equals the simple value ratio (index) that can be decomposed as follows:

\[ \sum \sum \alpha_{it} = \sum \sum \alpha_{i0} \]

b) Carli's index formula

In order to satisfy the commensurability axiom and to eliminate the structural component a price index should be a mean of ratios (relatives) rather than of a ratio of means.

\[ (1.2.4) \quad P_{0t}^{C} = \frac{1}{n} \sum p_{it} = \frac{1}{n} \sum a_{it} \quad \text{(price index of Carli 1764).} \]

\[ (1.2.5) \quad a_{it}^{\min} \leq P_{0t}^{C} \leq a_{it}^{\max} \]

c) Commensurability and time reversal test, choice of the type of average in aggregating price quotations

\[ (1.2.6) \quad P_{0t} = \frac{1}{P_{0t}} , \quad \text{time reversal test} \]

making a price index invariant to a change of the base period. Unfortunately many of the best index theoreticians are setting great store by this property

The emphasis placed on time reversibility as well as the idea that the formulas of Laspeyres and Paasche are "equally justified" (prompting the taking of an average of both formulas) rests on the tacit assumption that periods, 0 and \( t \) are having the same logical status. However, 0 and \( t \) are not equivalent and on the same footing. This can easily be seen as 0 is kept constant for some periods at least, while \( t \) denotes a sequence of periods, 1, 2, ...

In order to justify the rejection of Carli’s formula Haschka 1999 gave the following example:
**Price Relatives and Price Indices**

<table>
<thead>
<tr>
<th>i</th>
<th>( p_{0i} )</th>
<th>( p_{0t} )</th>
<th>( p_{it} )</th>
<th>( \text{price relatives} )</th>
<th>price index</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>10</td>
<td>0.5</td>
<td>Dutot ( P^D = 15/15 = 1 )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>200</td>
<td>2</td>
<td>Jevons ( P^J = \sqrt{0.5 \cdot 2} = 1 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \bar{p}_0 = 15 )</td>
<td>( \bar{p}_t = 15 )</td>
<td>Carli ( P^C = (2+0.5)/2 = 1.25 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

However

<table>
<thead>
<tr>
<th>i</th>
<th>( p_{0i} )</th>
<th>( p_{it} )</th>
<th>( \text{price relatives} )</th>
<th>price index</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200</td>
<td>210</td>
<td>1.05</td>
<td>Dutot ( P^D = 105/60 = 1.75 )</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>10</td>
<td>0.5</td>
<td>Jevons ( P^J = \frac{205}{200} \cdot 2 = 1 )</td>
</tr>
<tr>
<td></td>
<td>( \bar{p}_0 = 110 )</td>
<td>( \bar{p}_t = 110 )</td>
<td>Carli ( P^C = (2+0.5)/2 = 1.25 )</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 1.2.2:** Behavior of unweighted indices by type of mean

![Diagram showing behavior of unweighted indices](image-url)

<table>
<thead>
<tr>
<th>arithmetic (PA)</th>
<th>geometric (PG)</th>
<th>harmonic (PH)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.2.7) ( PA_{t0} &gt; (PA_{0t})^{-1} )</td>
<td>(1.2.8) ( PG_{t0} = (PG_{0t})^{-1} )</td>
<td>(1.2.9) ( PH_{t0} &lt; (PH_{0t})^{-1} )</td>
</tr>
<tr>
<td>Carli's index (1.2.4) ( p_{it}^C = \frac{\sum (p_{it}/p_{0i})}{n} )</td>
<td>Jevons' index (1.2.10) ( P_{it}^{JV} = \sqrt[\sum p_{it}]{\prod p_{0t}/p_{0i}} )</td>
<td>unnamed index (1.2.11) ( P_{0t}^{HM} = \frac{n}{\sum \frac{p_{0i}}{p_{it}}} )</td>
</tr>
<tr>
<td>( p_{it}^C &gt; 1/p_{0t}^C )</td>
<td>( p_{it}^{JV} )</td>
<td>( p_{0t}^{HM} )</td>
</tr>
</tbody>
</table>

These relations also hold for weighted means as well since they refer to the *kind of mean*. Thus for to the Laspeyres index (a PA-type) eq. 7 applies whereas for the Paasche index (PH-type) eq. 9 applies. For PHM see also fig. 2.2.1

(1.2.12) \( PA_{0t} = P_{0t}^C = 1/P_{0t}^{HM} \).

(1.2.13) \( P_{0t}^D = \sum \left( \frac{p_{0i}}{\sum p_{0i}} \right) \frac{p_{it}}{p_{0i}} \neq P_{0t}^C = \sum \left( \frac{1}{n} \right) \frac{p_{it}}{p_{it}}. \)

**Some new unweighted index functions**

The index

(1.2.12a) \( P_{0t}^{CSWD} = \sqrt{P_{0t}^C P_{0t}^{HM}} \) is known as CSWD (Carruthers – Sellwood – Ward - Dalen) index. And the index

(1.2.12b) \( P_{0t}^{HYB} = \sum \left( \frac{p_{0i}}{p_{it}} \right) \sqrt{\frac{p_{it}/p_{it}}{p_{0i}/p_{0i}}} = \sum \frac{p_{i}}{p_{0i}} \sqrt{(1/p_{0i}) \left(1/p_{it}/1/p_{0i}\right)} \)

has been introduced as "hybrid-index" by Jens Mehrhoff in a short note contributed to my book on Index Theory (p. 45 f.). He found the formula as a linear approximation of Jevons’ index.
index. \( P^{\text{HYB}} \) is also known as Balk-Walsh index because it corresponds as an unweighted index to the weighted Walsh-Index (eq. 2.2.9) and has been introduced by Bert Balk (2005).

Both indices, CSWD and HYB are special cases of a generalized average (see eq. 2.2.19). In sec. 2.2.b we will also mention some other index formulas, such as the exponential index.

Table 1.2.1: Some properties of unweighted means as indices

<table>
<thead>
<tr>
<th>commensurability (C)</th>
<th>time reversal test (T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>satisfied</td>
<td>Jevons ( P^{JV} ), CSWD and &quot;Hybrid&quot;-index</td>
</tr>
<tr>
<td>violated</td>
<td>Dutot ( P^{D} )</td>
</tr>
<tr>
<td>violated</td>
<td>Carli(^b) ( P^{C} )</td>
</tr>
<tr>
<td>violated</td>
<td>Drobisch ( P^{UD} )</td>
</tr>
</tbody>
</table>

\( a \) including Drobisch's unit value index; all index functions listed violate the factor reversal test
\( b \) the same is true for an unweighted harmonic mean.

Given that nowadays (an unwarranted) great store is set by the time reversal condition it is not surprising that in international standards for an unweighted aggregation of price quotations* the formulas \( P^{JV} \) and \( P^{D} \) are recommended while \( P^{C} \) (Carli) is banned.

* referring for example to the same commodity in different outlets

d) Stochastic and aggregative approach to index theory

<table>
<thead>
<tr>
<th>(old*) stochastic approach</th>
<th>aggregative approach*</th>
</tr>
</thead>
<tbody>
<tr>
<td>objective, inflation as the result of monetary factors (equally influencing all prices)</td>
<td>subjective, i.e. defined with reference to the expenditures (&quot;baskets&quot;) of specific consuming units (e.g. households)</td>
</tr>
<tr>
<td>unweighted means of price relatives (index conceived as [arithmetic, geometric etc] mean of a distribution of relatives)</td>
<td>derived from comparing expenditures (aggregates) in period t with those in the base period 0, consumer price index as a ratio of expenditures (of households)</td>
</tr>
</tbody>
</table>

* as opposed to new (see sec. 2.3), some authors distinguish unweighted (= old) and weighted (= new)

The formulas of Laspeyres and Paasche for example, can be interpreted in both ways, as (weighted) means of price relatives (in line with the stochastic approach) and as ratio of expenditures (as required in the aggregative approach). It should be noticed that none of the interpretations applies to the "ideal" index of Fisher or to all sorts of chain indices.

1.3. Index formulas of Laspeyres and Paasche

a) Dutot's index and Drobisch's unit value index  
b) Carli's index formula  
c) Commensurability and time reversal test  
d) Stochastic and aggregative approach

a) Price indices, dual interpretation

(1.3.1) \[ V_{0t} = \frac{\sum p_{it} q_{it}}{\sum p_{i0} q_{i0}} = \frac{\sum p_{i1} q_{i1}}{\sum p_{i0} q_{i0}} = \frac{P_{i1}}{P_{i0}} = \frac{q_{i1}}{q_{i0}} \] (value index).

(1.3.2) \[ P_{0t} = \frac{\sum p_{i0} q_{i1}}{\sum p_{i0} q_{i0}} = \frac{P_{i1}}{P_{i0}} \] (fixed budget index)
Both price indices, Laspeyres and Paasche can be interpreted in terms of "changing costs of a budget (basket)". It should be noted, however, that there is not a complete symmetry*: period 0 denotes a single, constant (for the "life time" of an index) period while period t is a variable period referring to many different periods, t = 1, 2 ...

* A symmetric interpretation of the Laspeyres and Paasche formula is also very popular in the case of the so-called "economic theory" of index numbers.

b) Price indices and quantity indices

Direct method by interchanging prices and quantities in the aggregative form of a price index Q = f(q0, p0, q1, p1) from P = f(p0, q0, p1, q1) .

(1.3.5) Q_{0t} = \sum b_{0t} \cdot w_i \quad (w_i = p_0 q_0 / \sum p_0 q_0)

Figure 1.3.1: Price and quantity indices

Methods to derive a quantity index

Direct method: interchanging of prices and quantities

Indirect method: division of V_{0t} by a price index

Independently defined quantity index

result: "cofactor", or "implied", or "indirect" quantity index", or "antithesis"

The formulas of Laspeyres and Paasche are related to the value index in the following manner

(1.3.6) V_{0t} = \frac{P_{0t}^L \cdot Q_{0t}^P}{P_{0t}^P \cdot Q_{0t}^L}

showing that Q_{0t}^P is the cofactor (or "factor antithesis") of P_{0t}^L and so is Q_{0t}^L to P_{0t}^P.

c) Asymmetry in the interpretation of the Laspeyres and Paasche formula

There is a remarkable difference between the two formulas, in particular with respect to:

- data requirements,
- an interpretation in terms of representativity and pure price comparison, and
• the underlying concept of measuring a price movement (rise or decline of prices).

Sequence of index numbers

\[ p_{01}, p_{02}, p_{03} \ldots : \frac{\sum p_1 q_0}{\sum p_0 q_0}, \frac{\sum p_2 q_0}{\sum p_0 q_0}, \frac{\sum p_3 q_0}{\sum p_0 q_0}, \ldots \]

\[ p_{p1}, p_{p2}, p_{p3} \ldots : \frac{\sum p_1 q_1}{\sum p_0 q_1}, \frac{\sum p_2 q_2}{\sum p_0 q_2}, \frac{\sum p_3 q_3}{\sum p_0 q_3} \]

The same consideration applies to series of successive Laspeyres- and Paasche quantity indices (which may result from deflation with \( P_{0t} \)): The Laspeyres indices (of prices or quantities) provide a result fully in line with the spirit of the principle of pure comparison, successive values are influenced only by the variable in question, that is prices or quantities respectively. Paasche indices, however, are always affected by both variables.

The following "antithetic" relationship was known already to Irving Fisher

\[ (1.3.7) \quad p_{L1}^{t0} = 1/p_{P1}^{t0} \quad \text{and} \quad (1.3.8) \quad p_{P1}^{t0} = 1/p_{L1}^{t0} \quad \text{("time antithesis").} \]

\[ (1.3.7a) \quad q_{L1}^{t0} = 1/q_{P1}^{t0} \quad \text{and} \quad (1.3.8a) \quad q_{P1}^{t0} = 1/q_{L1}^{t0}. \]

\[ (1.3.9) \quad p_{P1}^{t0} p_{L1}^{t0} = \sqrt{p_{P1}^{t0} p_{P1}^{t0}} \sqrt{p_{L1}^{t0} p_{L1}^{t0}} = \sqrt{p_{P1}^{t0} p_{P1}^{t0} \left(1/p_{P1}^{t0} \right) \left(1/p_{L1}^{t0} \right)} = 1. \]

**Table 1.3.1: Interpretation of the Laspeyres and Paasche price and quantity index formula**

<table>
<thead>
<tr>
<th>Price-index formula</th>
<th>Laspeyres</th>
<th>Paasche(^1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>numerator</td>
<td>imputed expenditures, i.e. expenditures as they were, if quantities were kept constant</td>
<td>empirical, i.e. observed actual (current period) expenditures referring to actual quantities (not to constant quantities)</td>
</tr>
<tr>
<td>denominator</td>
<td>empirical, i.e. actual base period expenditures (being constant); empirically observed and constant</td>
<td>imputed expenditures as they were, if prices were kept constant; measures of volume (as substitute for quantity)</td>
</tr>
<tr>
<td>time series</td>
<td>time series interpretation possible because only the numerator varies in ( p_{L1}^{t0}, p_{P1}^{t0}, p_{P3}^{t0} \ldots )</td>
<td>both, numerator and denominator vary, indices in a &quot;run&quot; ( p_{P1}^{t0}, p_{P2}^{t0}, p_{P3}^{t0} \ldots ), not comparable</td>
</tr>
<tr>
<td>price movement</td>
<td>directly measured: rising (descending) prices inferred from rising (decreasing) costs of a fixed budget(^2)</td>
<td>indirectly(^3) measured: rising prices because actual costs are higher as they were when prices remained constant</td>
</tr>
</tbody>
</table>

**Quantity-index formula**

| Concept of quantity movement | direct: quantities (volume) increased to the extent to which expenditure valued at constant prices has increased (i.e. rising volume) | indirect\(^3\): quantities are rising if value at current prices \( \Sigma p_t q_t \) is greater than \( \Sigma p_0 q_0 \). |

---

1 It is assumed that the Paasche price (quantity) index is used to measure price (quantity) level movement, but the real merits of the Paasche formula can be seen only in the case of deflation (see sec. 5.2).

2 a quantitatively fixed budget

3 indirect approach means: by comparing value and volume (both referring to actual consumption quantities).
d) Theorem of Ladislaus von Bortkiewicz concerning the relationship between Paasche and Laspeyres formulas

Denoting the price and quantity relative of an individual commodity $i$ by $a_{0t}^i$ and $b_{0t}^i$ we obtain the value ratio (relative) $c_{0t}^i = a_{0t}^i \cdot b_{0t}^i$ and by using weights $w_i = p_{0t}q_{0t}/\sum p_{0t}q_{0t}$

\[ P_{0t}^L = \sum a_{0t}^i w_i \quad (1.3.10) \quad Q_{0t}^L = \sum b_{0t}^i w_i \quad (1.3.11) \quad V_{0t} = \sum a_{0t}^i b_{0t}^i w_i \quad (1.3.12) \]

The covariance $C$ between (weighted) price and quantity relatives is given by

\[ C = \sum (a_{0t}^i - P_{0t}^L) (b_{0t}^i - Q_{0t}^L) w_i = V_{0t} - P_{0t}^L Q_{0t}^L = Q_{0t}^L (P_{0t}^P - P_{0t}^L) = P_{0t}^L (Q_{0t}^P - Q_{0t}^L), \]

or using $r_{ab}$, the correlation coefficient, and $V_a$, $V_b$ the coefficients of variation, we get

\[ C = V_{0t} - P_{0t}^L Q_{0t}^L = r_{ab} (s_a s_b) = r_{ab} P_{0t}^L Q_{0t}^L V_a V_b, \]

which is known as theorem of Bortkiewicz. It is often stated as follows

\[ \frac{V_{0t}}{P_{0t}^L Q_{0t}^L} = 1 + r_{ab} V_a V_b. \]

\[ \frac{P_{0t}^L}{P_{0t}^P} = \frac{Q_{0t}^L}{Q_{0t}^P} = 1 - \frac{C}{V_{0t}} = \frac{P_{0t}^L Q_{0t}^L}{V_{0t}} \quad (1.3.12a) \quad \frac{P_{0t}^P}{P_{0t}^L} = \frac{Q_{0t}^P}{Q_{0t}^L} = 1 + r_{ab} V_a V_b, \]

### Table 1.3.2: Relations between Laspeyres- and Paasche formulas

<table>
<thead>
<tr>
<th>Paasche</th>
<th>Laspeyres</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{0t} Q_{0t}$</td>
<td>$P^P Q^P = V^2/(V-C) &lt; V$ if $C &lt; 0$, $&gt; V$ if $C &gt; 0$</td>
</tr>
<tr>
<td>$P_{0t} P_{10}$</td>
<td>$P^P P^P = P_{0t}^P / P_{0t}^L &lt; 1$ if $C &lt; 0$</td>
</tr>
<tr>
<td>$P_{0t}^L P_{10}^L$</td>
<td>$P_{0t}^L P_{10}^L = P_{0t}^L / P_{0t}^P &gt; 1$ if $C &lt; 0$</td>
</tr>
</tbody>
</table>

The so called "Laspeyres-effect", that is the situation in which $P^L > P^P$ (and consequently also $Q^L > Q^P$) occurs when the price of commodity $i$ rises (that is $a_{0t}^i > 1$) the quantity tends to be reduced ($q_{0t}^i < q_{0t}$ hence $b_{0t}^i < 1$) and vice versa, that is we have a negative covariance $C$.

**Digression:**

**Critique of the Laspeyres index and the principle of "pure price comparison" by the US-Senate Advisory Commission (Boskin Commission, BC)**

The Laspeyres approach to price level measurement has often been criticized because of its fixed basket. There is a widespread belief that a "fixed basket" index overstates inflation (is "biased" upwards) and that a chain index, or any other index giving a higher weight to the more recent consumption pattern will do a better job and will result in a lower inflation rate. Furthermore advocates of the "economic theory" (or true cost of living index = COLI) approach also heavily criticize the Laspeyres principle because the point of reference should not be the same quantity of products in period $t$ and $0$ but rather the same utility.

Such ideas were vigorously advanced by an Advisory Commission (also known as Boskin Commission [BC] because it was headed by Michael Boskin) to the US-Senate Commission of Finance\(^1\) to

---

\(^1\) It is worth noticing that the installation of the BC should be seen against the background of making extensively use of indexation and thus having a problem with budget and debt management in the USA, calling for a cut in
explore a possible upwards bias of the US Consumer Price Index (CPI, then a traditional fixed basket Laspeyres index). The BC published an interim report in 1995 and a final report in 1996 titled "Toward a More Accurate Measure of the Cost of Living"\(^2\) - both widely made public in the internet - from which subsequently some quotations are taken.

**Table 1.3.3:** Examples for Boskin Commission's suggestions for improvement of the CPI

<table>
<thead>
<tr>
<th></th>
<th>price quotations in traditional CPI</th>
<th>Boskin Commission's proposal</th>
</tr>
</thead>
<tbody>
<tr>
<td>medical care*</td>
<td>expensive surgical operations in the case of heart attacks or ulcer</td>
<td>treating heart attacks or ulcer with generic drugs</td>
</tr>
<tr>
<td>entertainment</td>
<td>visits to cinemas and theatres</td>
<td>rent of videos, seeing a movie at home</td>
</tr>
<tr>
<td></td>
<td>buying old-fashioned bound-book versions of encyclopaedias</td>
<td>CD-ROM encyclopaedias, surfing the internet, going to libraries</td>
</tr>
</tbody>
</table>

* Interestingly substitutions to maintain a constant level of utility may occur across borders of a commodity classification: for example services (surgical operations) are substituted by goods (generic drugs), and vice versa (buying books vs. renting books).

Critique of the "utility" reasoning in the COLI-approach:

1. The distinction between inflation and welfare measurement becomes blurred, questionable imputations of gains or losses in utility are instigated, and 2. the notion of "good" becomes boundless, and finally 3. we move away from statistics of observable phenomena to speculations about levels of utility or a "fair" amount of income necessary for a "compensation".

**Figure 1.3.2: Conceptual differences in price level measurement**

<table>
<thead>
<tr>
<th>Price index should measure according to</th>
</tr>
</thead>
<tbody>
<tr>
<td>the Laspeyres' formula = COGI</td>
</tr>
<tr>
<td>the general concept</td>
</tr>
<tr>
<td>the change of expenditure needed to</td>
</tr>
<tr>
<td>buy a constant (with respect to kind and quantities of goods) &quot;basket&quot;</td>
</tr>
<tr>
<td>objective of measurement</td>
</tr>
<tr>
<td>&quot;pure&quot; comparison of prices of a well defined set of commodities</td>
</tr>
<tr>
<td>why adjustment for quality change?</td>
</tr>
<tr>
<td>to ensure comparability of prices over time (to isolate a genuine increase from a rise due to quality improvement)</td>
</tr>
</tbody>
</table>

expenditures for social purposes in these days. Hence there was most obviously a certain political interest in getting a lower rate of inflation. It is interesting (and depressing) to note that given this political background there were many theoreticians and statisticians who readily agreed in a unanimous critique of the Laspeyres formula as overstating inflation and who worked out estimates (then sometimes called “guestimates”) of the amount of bias in the US-CPI. On the other hand those defending Laspeyres' index formula were few if any.

\(^2\) The title was misleading because the concern was a new conceptual basis of measurement, rather than how to observe and measure more accurately price movement on the basis of a given generally accepted way to measure inflation.
With a little imagination we will always find some additional sources of utility. The US-CPI had been criticized by the BC for example for not taking into account "the benefit to consumers of being able to keep track more easily of children, spouses, or of aged parents" in reporting prices of cellular telephones. As if this kind of benefits has to be seen as equivalent to offering telephones at a lower price.

Chapter 2 Approaches to index theory

2.1. Outline of index theories/approaches

Figure 2.1.1: Some "constructive" approaches in index theory

1) shading of boxes indicates that this index is an "ideal index" (i.e, satisfying the factor reversal test).
2) factorial approach.
3) these indices are not derived from a model of decomposing the value change.
4) as an example: Cobb-Douglas index (transitivity holds) or refined Törnqvist approaches in order to come closer to factor reversibility (Theil, Sato).

2.2. Irving Fisher's mechanistic approach and reversal tests

a) Fisher's systematic search for formulas  d) A weak variant of the time reversal test
b) Generalization of means  e) Fisher's philosophy, the circular test
c) Fisher's reversal tests, crossing of formulas

a) Fisher's systematic search for formulas

Fisher introduced four methods of weighting price relatives:

| I | base weights: \( p_0 q_0 \) (Laspeyres) |
| II | real expenditures: \( p_t q_0 \) "hybrid" |
| III | pure price movement: \( p_t q_0 \) "hybrid" |
| IV | current weights: \( p_t q_t \) (Paasche) |

and he combined them with six types of means. The result is a collection of "second generation" index-formulas such as \( P_L, P^P, P^{AH}, P^{PA}, P^{HH}, \) and \( P^{HH} \) (fig. 2.2.1).
**Figure 2.2.1:** Family tree of index formulas (according to Köves)

### First generation indices (unweighted indices)

<table>
<thead>
<tr>
<th>additive structure</th>
<th>geometric block see tab.2.2.1</th>
<th>positional parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 aggregative</td>
<td>2 arithmetic</td>
<td>3 harmonic</td>
</tr>
<tr>
<td>4 geometric mean</td>
<td>5, 6 median and mode</td>
<td></td>
</tr>
<tr>
<td>Dutot $P^D_{0t}$</td>
<td>Carli $P^C_{0t}$</td>
<td>Jevons $P^J_{0t}$</td>
</tr>
</tbody>
</table>

### Second generation indices (weighted indices)

<table>
<thead>
<tr>
<th>means</th>
<th>I: base $p_0q_0$</th>
<th>II: hybrid $p_0q_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>arithmetic</td>
<td>$P^L_{0t}$ Laspeyres</td>
<td>$P^P_{0t}$ Paasche</td>
</tr>
<tr>
<td>harmonic</td>
<td>$P^{HB}_{0t}$ see eq. 2.2.2</td>
<td>$P^{HH}_{0t}$</td>
</tr>
<tr>
<td>geometric</td>
<td>less well known formulas, see text below and table 2.2.1</td>
<td></td>
</tr>
</tbody>
</table>

Note that some of the combinations are identical due to inherent relations between the arithmetic and the harmonic mean: both index formulas, $P^L$ and $P^P$ can be expressed in two ways, using pure and hybrid weights.

### Third generation indices see tab. 2.2.1 (derived by crossing)

- **crossing** (averaging) of preferred mean: arithmetic, harmonic, geometric
- **formulas** (preferred: $P^L_{0t}$, $P^P_{0t}$) formulas of Drobisch, Fisher etc.
- **weights** (preferred I, IV) formulas of Marshall/Edgeworth, Walsh, etc.

* "Crossing" in a more specific use refers to averaging an index $P$ with its "antithesis". If this is done using a geometric mean Fisher called it "rectifying".

\[
\begin{align*}
(2.2.1) & \quad P^P_{0t} = \sum p_i \frac{p_i q_i}{p_0} = \frac{\sum p_i \tilde{q}_i}{\sum p_i q_i} > P^L_{0t} \quad \text{where } \tilde{q}_i = \frac{q_i}{P_{0t}} \quad \text{(R. H. I. Palgrave, 1886)} \\
(2.2.2) & \quad P^{HB}_{0t} = \frac{\sum p_0 q_0}{\sum (p_0 q_0 / p_t)} \leq P^L_{0t} \\
(2.2.3) & \quad P^{CD}_{0t} = \prod \left( \frac{p_i}{p_0} \right)^{\alpha_i} = \prod (a_{i0})^{\alpha_i} \quad \text{(CD = Cobb-Douglas)}
\end{align*}
\]

The name refers to the well known Cobb-Douglas (production) function.

where the $\alpha_i$-coefficients are any real valued arbitrary weights not further specified, except for $\alpha_i \geq 0$ and $\sum \alpha_i = 1$. Consider two factors, $x_1$, $x_2$ only. It is easy to see that the geometric mean of these values, that is $\bar{x}_G = \sqrt{x_1 \cdot x_2}$ will be the geometric mean of the arithmetic and the harmonic mean, thus
(2.2.4) \( \bar{x}_G = \sqrt{\bar{x} \cdot \bar{x}_H} \) which is the reason for the relation between \( P^F \), \( P^{HPL} \) and \( P^{DR} \).

The harmonic mean of quantities is defined as
\[
q_{it} = \frac{2(q_{i0} q_{it})}{q_{i0} + q_{it}}
\]
(Geary-Kahmis method, \( \to \) ch. 8).

Table 2.2.1: Some "log-change-indices" derived from Fisher’s scheme to systematize index formulas, geometric means and types of weights

<table>
<thead>
<tr>
<th>weights I: ( w_{i0} = p_0 q_0 / \sum p_0 q_0 )</th>
<th>weights IV: ( w_{it} = p_i q_i / \sum p_i q_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>logarithmic Laspeyres index</td>
<td>logarithmic Paasche* index</td>
</tr>
<tr>
<td>( DP^{L}<em>{oi} = \prod \left( \frac{p</em>{it}}{p_{i0}} \right)^{w_{i0}} )</td>
<td>( DP^{P}<em>{oi} = \prod \left( \frac{p</em>{it}}{p_{i0}} \right)^{w_{it}} )</td>
</tr>
</tbody>
</table>

geometric mean of both indices: Törnqvist index
\[
P^T_{oi} = \sqrt{DP^{L}_{oi} DP^{P}_{oi}} = \prod (p_{i0} / p_{oi})^w \text{ where } w = \frac{1}{2}(w_0 + w_1) \text{ or } \ln(P^T_{oi}) = \frac{1}{2} [\ln(DP^{L}_{oi}) + \ln(DP^{P}_{oi})]
\]

There are similarities between \( P^F \) and \( P^P \) on the one hand and \( P^L \) and \( DP^L \), \( DP^P \) on the other hand.

* Obviously the construction of \( DP^P \) suggests the name "logarithmic Palgrave" rather than "logarithmic Paasche". Yet it is consistent to refer to Paasche in this context as shown in sec. 3.4.

\( P^{ME} \) makes use of the arithmetic mean of weights, \( \frac{1}{2}(q_{i0} + q_{it}) \) and it also takes the form of a weighted mean of \( P^L \) and \( P^P \): (2.2.8a)
\[
P^{ME}_{oi} = p^{ME}_{oi} = \frac{P^L_{oi} + V_{oi}}{1 + Q^L_{oi}} = \left( \frac{1}{1 + Q^L_{oi}} \right) P^L_{oi} + \left( \frac{Q^L_{oi}}{1 + Q^L_{oi}} \right) P^P_{oi}
\]

(2.2.11) \[
\frac{\sum p_i q_i}{\sum p_0 q_0} \left[ \frac{\sum p_0 q_0 (q_0 + p_i)}{q_i} \frac{\sum p_i q_i (q_i + p_0)}{p_0} \right]^{1/4}
\]

 Legacy of Irving Fisher

Table 2.2.2: Some well known formulas derived from crossing\(^1\) formulas and weights

<table>
<thead>
<tr>
<th>arithmetic mean</th>
<th>geometric mean</th>
<th>harmonic mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>crossing Laspeyres and Paasche index formula</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2.2.5) Droebisch 1871 (or Sidgwick 1883) ( P^{DR}<em>{oi} = \frac{1}{2}(P^L</em>{oi} + P^P_{oi}) )</td>
<td>(2.2.6) Irving Fisher 1922 ( P^F_{oi} = \sqrt{P^{L}<em>{oi} P^{P}</em>{oi}} ) = &quot;ideal index&quot; of Fisher</td>
<td>(2.2.7) no-name index(^2) (not in use), to be called ( P^{HPL}<em>{oi} = \left( P^F</em>{oi} \right)^2 / P^{DR}_{oi} )</td>
</tr>
<tr>
<td>crossing Laspeyres and Paasche weights</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2.2.8) Marshall and Edgeworth 1887 ( P^{ME}_{oi} = \frac{\sum p_i (q_0 + q_i)}{\sum p_0 (q_0 + q_i)} )</td>
<td>(2.2.9) Walsh 1901 ( P^{W}_{oi} = \frac{\sum p_i \sqrt{q_0 q_i}}{\sum p_0 \sqrt{q_0 q_i}} )</td>
<td>(2.2.10) Geary-Khamis(^3) ( P^{GK}_{oi} = \frac{\sum p_i q_0 q_i}{\sum p_0 (q_0 q_i)} )</td>
</tr>
</tbody>
</table>

\(^1\) As the Paasche formula is the time and factor "antithesis" of Laspeyres and vice versa, we may say, second generation indices are derived from crossing a formula with her antithesis instead of crossing two formulas.
2) The name, given here for convenience of presentation, should be "Harmonic-Paasche-Laspeyres" (HPL).
3) Result of applying the method of Geary and Khamis for multinational comparisons to m = 2 countries only, A and B. The periods 0 and t are taking in eq. 10 the place of the countries A and B.

**Figure 2.2.2:** Interpretations of second and third generation indices

<table>
<thead>
<tr>
<th>second generation</th>
<th>Laspeyres (P_L), Paasche (P_P), Palgrave (P_PA), log. Lasp. (DP_L)</th>
</tr>
</thead>
<tbody>
<tr>
<td>third generation</td>
<td>Drobisch (P_DR), Fisher (P_F), Marshall-Edgeworth (P_ME), Walsh (P_W), Geary-Khamis (P_GK)</td>
</tr>
</tbody>
</table>

**Admissible interpretations**

<table>
<thead>
<tr>
<th>mean of individual price relatives</th>
<th>ratio of aggregates (expenditures, revenues)</th>
<th>mean of index formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>applies to all second generation indices</td>
<td>applies to (= yes) P_L, P_P; with reference to q_0, q_t; yes (average quantities): P_ME, P_W, P_GK, no: P_DR, P_F</td>
<td>P_L: ( \alpha = 1 ), P_P: ( \alpha = 0 ), P_PA farfetched 4); no: DP_L; P_DR, P_F yes (( \alpha = \frac{1}{2} )), P_ME yes; however P_W, P_GK no</td>
</tr>
</tbody>
</table>

1) that is numerator and denominator can be expressed (regarded) as sums of expenditures (products of prices and certain quantities q*, not necessarily the same in numerator and denominator).

2) this means: the index function can be expressed as a special case of the general (weighted) arithmetic mean \( \alpha P_L + (1-\alpha) P_P \) or of the general (weighted) geometric mean \( (P_L)^\alpha + (P_P)^{1-\alpha} \).

3) interpretation possible by kind of construction of this type of indices (as they are derived as means)

4) quantities in the numerator, q_t (p_t/p_0) might be viewed as “adjusted” quantities q. The situation is similar in cases, like \( P_AH, P_HB \) and \( P_HH \) in fig. 2.2.1.

**b) Generalization of means**

The unweighted arithmetic mean is a special case of the weighted linear combination (or weighted arithmetic mean, or convex combination)

\[ P_{0t}^{CC}(\alpha) = \alpha P_L + (1-\alpha) P_P \]

Depending on the parameter \( \alpha \) we get:

\[ P_{0t}^{CC}(\alpha = 0) = P_{0t}^P \quad P_{0t}^{CC}(\alpha = \frac{1}{2}) = P_{0t}^{DR} \quad P_{0t}^{CC}(\alpha = 1 + Q_L^{-1}) = P_{0t}^{ME} \quad P_{0t}^{CC}(\alpha = 1) = P_{0t}^L \]

The generalized **weighted geometric mean** \( P_{0t}^{GM} \)

\[ P_{0t}^{GM}(\alpha) = \left( P_{0t}^P \right)^\alpha \cdot \left( P_{0t}^P \right)^{1-\alpha} \]

known also as **generalized Fisher index**. Depending on the parameter \( \alpha \) we get

\[ P_{0t}^{GM}(\alpha = 0) = P_{0t}^P \quad P_{0t}^{GM}(\alpha = \frac{1}{2}) = P_{0t}^F \quad P_{0t}^{GM}(\alpha = 1) = P_{0t}^L \]

It follows from above \( P_{0t}^P \leq P_{0t}^F \leq P_{0t}^L \) or \( P_{0t}^P \geq P_{0t}^F \geq P_{0t}^L \).

Likewise \( P_{0t}^{DR} \) (Drobisch), and \( P_{0t}^{HPL} \) (harmonic mean) will always lie in the interval bounded by \( P_{0t}^P \) and \( P_{0t}^L \) (where normally \( P_{0t}^P \) will be the lower bound and \( P_{0t}^L \) the upper bound).
The generalized (weighted) harmonic crossing of formulas is

$$(2.2.14) \ P_{0t}^H(\alpha) = \frac{P_{0t}^L P_{0t}^P}{\alpha P_{0t}^L + (1 - \alpha) P_{0t}^P}, \ \alpha = \frac{1}{2} \rightarrow P_{0t}^{HPL} \text{ has no relevance.}$$

$\text{P}^{CC}(\alpha),$ and $\text{P}^H(\alpha)$ respectively (where $\alpha = \frac{1}{2}$) both violate the time reversal test and are "time antithesis" of one another, however $\text{P}_{0t}^{GM}(\alpha) = \frac{1}{P_{0t}^{GM}(\alpha)}$ and $\text{P}_{0t}^{GM}(\alpha) \cdot Q_{0t}^{GM}(\alpha)$ differs from $V_{0t}$ if $\alpha \neq \frac{1}{2}$. Hence

All indices of the generalized Fisher index type ($0 < \alpha < 1, \ \alpha \neq 0 \text{ and } \alpha \neq 1$) will
- satisfy the time reversal test, but
- fail the factor reversal test, except for the "ordinary" Fisher index (e.g. the case $\alpha = \frac{1}{2}$).

It can be shown that not only crossing of formulas but also crossing of weights $q_0$ and $q_t$ leads to formulas with values that lie in the interval $[P^L, P^P]$.

The concept of a **power mean** (or *moment mean*) $\bar{x}_r(\alpha)$ of degree $r$ with weights (relative frequencies) $h_1, h_2, ..., h_m$ in statistics is defined as follows:

$$(2.2.17) \ \bar{x}_r(\alpha) = \left[ \frac{1}{r} \sum h_i \left( \frac{x_i}{P_{0t}} \right)^r \right]^{1/r}, \text{ or with weights } w_i \text{ and price relatives}$$

$$(2.2.17a) \ P_{0t}^{PM}(r) = \left[ \sum w_i \left( \frac{p_i}{p_{0t}} \right)^r \right]^{1/r}. \text{ Example } r = 2$$

$$(2.2.18) \ P_{0t}^{QM} = P_{0t}^{PM}(r = 2) = \sqrt{\frac{\sum p_i^2}{\sum p_0 q_i}} \text{ (quadratic mean index).}$$

**Products of power means** like $\bar{x}_r(\alpha) \cdot \bar{x}_p(\alpha)$ especially if $k = -r$ give

$$(2.2.19) \ P_{0t}^{pp}(r) = \left[ \sum s_{i0} \left( \frac{p_i}{P_{0t}} \right)^{r/2} \right]^{1/r} \cdot \left[ \sum s_{it} \left( \frac{p_i}{P_{0t}} \right)^{-r/2} \right]^{-1/r}$$

where $s_{i0} = \frac{P_{0t} q_{i0}}{\sum P_{0t} q_{i0}}$ and $s_{it} = \frac{P_{it} q_{it}}{\sum P_{it} q_{it}}$ are expenditure shares. Again Fisher's ideal index is a special case ($r = 2$). Another special case is $r = 1$

$$(2.2.20) \ P_{0t}^{pp}(r = 1) = V_{0t} / Q_{0t}^W \text{ where } Q_{0t}^W = \frac{\sum q_i \sqrt{P_{0t} P_i}}{\sum q_0 \sqrt{P_{0t} P_i}}$$

which is the Walsh type cofactor price index (or factor antithesis of the quantity index $Q^W$).

In the unweighted case, that is $s_{i0} = s_{it} = 1/n$ we get

---

3 Strictly speaking the following formula is not a product of power means (because of the exponents $r/2$ and $-r/2$ instead of $r$ and $-r$). The formula (2.2.19) is also known as **quadratic mean** (a term, however, also used for eq. 2.2.18).
Summary table regarding products of power means (quadratic mean)

<table>
<thead>
<tr>
<th>$P_{0t}^{pp}$ (r)</th>
<th>weighted (2.2.19)</th>
<th>unweighted ($s_0 = s_t = 1/n$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = 1$</td>
<td>$V_{ot}/Q_{0t}^W$ Walsh co-factor price index</td>
<td>Hybrid index $P_{0t}^{hyb} = \sum \sqrt{p_{at}/p_{0t}}$</td>
</tr>
<tr>
<td>$r = 2$</td>
<td>Fisher's ideal index $P_{0t}^{F}$</td>
<td>CSWD index $P_{0t}^{CSWD} = \sqrt{\sum (p_{at}/p_{0t})}$</td>
</tr>
</tbody>
</table>

Other means in index theory

(2.2.21) $\text{Ex}(x) = \text{Ex}(x_1, \ldots, x_n) = \ln \left[ \frac{1}{n} \sum \exp(x_i) \right]$

is known as (unweighted) exponential mean. With weights $h_i$, it reads as follows

(2.2.21a) $\text{Ex}(x) = \text{Ex}(x_1, \ldots, x_n) = \ln \left[ \sum \exp(x_i) h_i \right]$ where $\sum h_i = 1$.

The logarithmic mean of two variables, $x_1$ and $x_2$ (introduced already in sec.1.1) is given by

(2.2.22) $\ln \left( \frac{x_1 - x_2}{\ln(x_1/x_2)} \right) = \ln \left( \frac{x_2 - x_1}{\ln(x_2/x_1)} \right)$ (defined for two values $\{x_1$ and $x_2\}$ only).

c) Fisher’s reversal tests, "crossing" and "rectifying" of formulas

The motivation to require the much stronger, and highly restrictive factor reversal test instead of the product test is rarely if ever spelled out in detail. It seems to be the desire to do both, inflation measurement and deflation with the help of the same price index.

The product test does not require $P$ and $Q$ to have formulas of the same structure.

Fisher used the notion of "time antithesis" $T(P)$ of an index $P$, and "factor antithesis" of the quantity index $Q$ (that is $P = F(Q)$ which is a price index) and the concept of a double antithesis (see fig. 2.2.3). For example the Laspeyres formula is the "time antithesis" and also the "factor antithesis" of the Paasche formula and vice versa Another idea of Fisher was to cross (average) a price index $P$ (or a quantity index $Q$) to one of its three antithesis in order to find a new price index formula. A special relationship exists in the case of a geometric mean crossing (= rectification) for any $P$ in that

(2.2.26) $P^* = \sqrt{P \cdot T(P)}$ meets time reversal test, $P^*_{0t} = 1/P^*_{0t}$, and

(2.2.27) $P^* = \sqrt{P \cdot F(Q)}$ along with (2.2.27a) $Q^* = \sqrt{Q \cdot F(P)}$

is factor reversible: $P^* Q^* = V$.

The following relationships hold for the antithesis relation:

1. they are reflexive $T(T(P)) = P$, and $F(F(P)) = P$, and

2. the symmetry between time - and factor antithesis relation $T(F(P)) = F(T(P))$

Using equation 2.2.27 it is always possible to construct a pair of index formulas $P^*$ and $Q^*$ conforming with factor reversibility. Often, however, it is difficult enough to find a meaningful interpretation for such index formulas, like $P^*$ and $Q^*$. For example the index
(2.2.28) \[ Q^{GK}_{0t} = \frac{\sum q_i \left( \frac{p_0 p_i}{p_0 + p_i} \right)}{\sum q_i \left( \frac{p_0 p_i}{p_0 + p_i} \right)} \] is the direct quantity index of \( P^{GK}_{0t} \) and the geometric mean of its factor antithesis and \( P^{GK}_{0t} \) is
\[
P^{(GK)}_{0t} = \left[ \frac{\sum p_i q_i \left( \frac{q_0 q_i}{q_0 + q_i} \right)}{\sum p_i q_i \left( \frac{q_0 q_i}{q_0 + q_i} \right)} \right].
\]

The \( Q^* \) index derived according to eq. 2.2.27a) is in the case of the GK index
\[
Q^{(GK)}_{0t} = \left[ \frac{\sum p_i q_i \left( \frac{p_0 p_i}{p_0 + p_i} \right) \sum p_0 \left( \frac{q_0 q_i}{q_0 + q_i} \right)}{\sum p_i q_i \left( \frac{p_0 p_i}{p_0 + p_i} \right) \sum p_0 \left( \frac{q_0 q_i}{q_0 + q_i} \right)} \right],
\]

obviously \( P^{(GK)}_{0t} Q^{(GK)}_{0t} = V_{0t} \)
\[
P_{0t}^F = \sqrt{P_{0t}^L P_{0t}^P} = \frac{1}{P_{0t}^F} \text{ and } V_{0t} = D_{0t}^F Q_{0t}^F = P_{0t}^F \sqrt{Q_{0t}^L Q_{0t}^P} = \sqrt{\frac{P_{0t}^L Q_{0t}^P}{V_{0t}} V_{0t} \frac{P_{0t}^L Q_{0t}^P}{V_{0t}}}. \]

Index numbers with weights obtained by crossing of weights (e.g. quantity weights \( q_{0i} \) and \( q_{it} \) in a price index) such as the index formulas of Edgeworth-Marshall (\( P^{ME} \)), Walsh (\( P^{W} \)) or Geary Kahmis (\( P^{GK} \)) always satisfy the time reversal test (since the weights are the same whatever the base period may be, 0 or \( t \)), but not necessarily the factor reversal test.

The same is true for index numbers using any constant weights (the same in numerator and denominator of course) not depending on periods \( 0 \) and \( t \) (like Lowe's prices index, or the Cobb Douglas index).

Crossing of formulas, for example: \( P_{0t}^{DR} = \left( P_{0t}^L + P_{0t}^P \right)/2 \) does not meet the time reversal, and the factor reversal condition either. We get, not surprisingly \( P_{0t}^{DR} = \frac{1}{P_{0t}^{HPL}} \neq \frac{1}{P_{0t}^{DR}} \),

showing that the harmonic \( P^{HPL} \) - index and the arithmetic Droebisch index are a pair of time antithetic indices like the arithmetic Laspeyres and the harmonic Paasche index.

**Figure 2.2.3:** Crossing \( P \) with an antithesis as finder of formulas

<table>
<thead>
<tr>
<th>crossing</th>
<th>average of ( P ) and an antithesis of ( P )</th>
<th>special case: rectifying</th>
</tr>
</thead>
<tbody>
<tr>
<td>time antithesis</td>
<td>( P = T(P) )</td>
<td>take the (unweighted) geometric mean</td>
</tr>
<tr>
<td>factor antithesis</td>
<td>( P = F(Q) )</td>
<td></td>
</tr>
<tr>
<td>double</td>
<td>( F(T(Q)) = T(F(Q)) )</td>
<td></td>
</tr>
</tbody>
</table>

**time** (2.2.23) \( P_{0t}^* = T(P_{0t}) = 1/P_{0t} \)

\( P^* \) is the time antithesis of \( P \)

\( P^* \) is the time antithesis of \( P \) since \( P_{0t}^L = 1/P_{0t}^p \) and vice versa \( P^p = T(P^L) \)

**factor** (2.2.24) \( P_{0t}^* = F(Q_{0t}) = V_{0t}/Q_{0t} \)

\( P^* \) is the factor antithesis of \( Q \); accordingly \( F(P_{0t}) \), the factor antithesis of \( P_{0t} \), is a quantity index
due to \( V_{0t} = P_{0t}^L Q_{0t}^P = P_{0t}^P Q_{0t}^L \) we have: \( P^L = F(Q^p) \) and vice versa \( P^p = F(Q^L) \)

**double** (2.2.25) \( P_{0t}^* = F(T(Q)) = T(F(Q)) \)

\( P^* \) is the double antithesis of \( Q \)

\( P^* \) is the double antithesis of \( Q^L \)

\( P^* \) is the double antithesis of \( Q^p \)
Figure 2.2.4: Time and factor reversal test of second generation indices (additive block only)

<table>
<thead>
<tr>
<th>weighting schemes: I = p₀q₀ ; II: p₀qₜ ; III: pₜq₀ and IV: pₜqₜ</th>
</tr>
</thead>
<tbody>
<tr>
<td>arithmetic means (A-) and weights (I → IV)</td>
</tr>
<tr>
<td>A - I: Pₐₙ₀t     A - II: Pₐₙ₀t     A - III: Pₐₙ₀t    A - IV: Pₐₙ₀t *</td>
</tr>
<tr>
<td>time antithesis</td>
</tr>
<tr>
<td>harmonic means (H-) and weights (IV → I)</td>
</tr>
<tr>
<td>H - IV: Pₐₙ₀t    H - III: Pₐₙ₀t    H - II: Pₐₙ₀t    H - I: Pₐₙ₀t</td>
</tr>
<tr>
<td>factor antithesis</td>
</tr>
<tr>
<td>Pₐₙ₀t → Qₐₙ₀t = Qₐₙ₀t   Pₐₙ₀t → Qₐₙ₀t = Qₐₙ₀t</td>
</tr>
<tr>
<td>formulas not useful</td>
</tr>
</tbody>
</table>

* PA = Palgrave

The indices of Geary - Khamis, Walsh and Marshall - Edgeworth pass the time reversal but not the factor reversal test. It can easily be seen that for example indices of Walsh and Marshall - Edgeworth fail the factor reversal test:

\[ P_{0t}^W \cdot Q_{0t}^W = \frac{\sum p_1 \sqrt{q_0 q_1}}{\sum p_0 \sqrt{q_0 q_1}} \cdot \frac{\sum q_1 \sqrt{p_0 p_1}}{\sum q_0 \sqrt{p_0 p_1}} \neq V_{0t}. \]

Likewise

\[ P_{0t}^{ME} Q_{0t}^{ME} = V_{0t} \left( \frac{1 + \frac{1}{Q^P}}{1 + \frac{1}{P^F}} \right) \left( 1 + P^L \right) \left( 1 + P^L \right) \neq V_{0t}. \]

According to the theorem of L. v. Bortkiewicz we have

\[ p_{0t}^{PL} p_{10}^{PL} = \frac{P^L Q^L}{V} = \frac{P^L}{P^F} = 1 - \frac{C}{V} \quad \text{and} \quad \frac{p_{0t}^{PF} p_{10}^{PF}}{V} = \frac{P^P Q^P}{V} = \frac{P^P}{P^F}, \]

and we also see that

\[ \frac{P^{DR} Q^{DR}}{V} = -1 = \frac{1}{2} \left( \frac{1}{2} \frac{P^L Q^L}{V} - 1 \right) + \frac{1}{2} \left( \frac{P^P Q^P}{V} - 1 \right) \]

the relative deviations of \( P^L \) and \( P^P \) are averaged.

Table 2.2.3: Factor reversal test in the case of third generation indices

<table>
<thead>
<tr>
<th>name of index</th>
<th>(direct) quantity index</th>
<th>cofactor quantity index</th>
<th>TR</th>
<th>FR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crossing of formulas ( P^L ) and ( P^P )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Drobisch</td>
<td></td>
<td></td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Fisher</td>
<td></td>
<td></td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>harmonic (HPL)</td>
<td></td>
<td></td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Crossing of weights*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Marshall - Edgeworth</td>
<td></td>
<td></td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Walsh</td>
<td></td>
<td></td>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>

* as for the Geary - Khamis formula see equations 2.2.28 and 28a
d) A weak variant of the time reversal test

It is obviously rather restrictive to require an index $P_{t0}$ to be the inverse index $P_{0t}$. It appears sufficient to postulate:

\begin{equation}
(2.2.29) \quad \text{if } P_{0t} > 1 \text{ then } P_{t0} < 1 \text{ and if } P_{0t} < 1 \text{ then } P_{t0} > 1.
\end{equation}

This requirement seems to be reasonable and not too ambitious: it is only desired that an increase in the direction $0 \rightarrow t$ should correspond to a decline in the opposite direction $t \rightarrow 0$ and vice versa.

**Example 2.2.2** Assume the following prices and quantities

<table>
<thead>
<tr>
<th>$i$</th>
<th>$p_{i0}$</th>
<th>$p_{it}$</th>
<th>$q_{i0}$</th>
<th>$q_{it}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>15</td>
<td>80</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>18</td>
<td>10</td>
<td>80</td>
</tr>
</tbody>
</table>

Calculate the following indices $P^C$ (Carli), $P^L$, $P^P$, $P^{DR}$ (Drobisch), each in both directions, that is $0 \rightarrow t$ and $t \rightarrow 0$. The results are as follows:

<table>
<thead>
<tr>
<th>formula</th>
<th>$P_{t0}$ direction $0 \rightarrow t$</th>
<th>$P_{t0}$ direction $t \rightarrow 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carli</td>
<td>$P^C_{0t} = (1.25 + 0.9)/2 = 1.075 &gt; 1$</td>
<td>$P^C_{t0} = 0.9555 &lt; 1$</td>
</tr>
<tr>
<td>Laspeyres</td>
<td>$P^L_{0t} = 1380/1160 = 1.1897 &gt; 1$</td>
<td>$P^L_{t0} = 1/P^L_{0t} = 1.0575 &gt; 1$</td>
</tr>
<tr>
<td>Paasche</td>
<td>$P^P_{0t} = 1740/1840 = 0.9457 &lt; 1$</td>
<td>$P^P_{t0} = 1/P^P_{0t} = 0.8406 &lt; 1$</td>
</tr>
<tr>
<td>Drobisch</td>
<td>$P^{DR}_{0t} = 1.0677 &gt; 1$</td>
<td>$P^{DR}_{t0} = 0.9490 &lt; 1$</td>
</tr>
</tbody>
</table>

Thus both, the Laspeyres- as well as the Paasche formula may fail this weak time reversal test, while the indices of Carli and Drobisch (or Sidgwick) will pass this test *necessarily* (though both indices do *not* satisfy the time reversal test).

♦

e) Fisher's philosophy in evaluating formulas by reversal tests and the circular test

Fisher's *seven point scale* ranging from worthless to superlative:

| 1. worthless, 2. weak, 3. correct, 4. good, 5. very good, 6. excellent and 7. superlative |
| examples: 5: Laspeyres and Paasche, 6: equation 2.2.11, 7: the two crossed indices $P^F$ and $P^{ME}$ |

The meaning and significance of Fisher's circular test

\begin{equation}
(2.2.30) \quad P_{01} P_{12} = P_{02} , \text{ and in connection with identity}
\end{equation}

\begin{equation}
(2.2.30a) \quad P_{01} P_{12} P_{20} = P_{00} = 1,
\end{equation}

\begin{equation}
(2.2.30b) \quad P_{01} P_{12} P_{23} = P_{03} .
\end{equation}

\begin{equation}
(2.2.31) \quad P^L_{0t} = \frac{\sum p_i q}{\sum p_0 q} \quad \text{(Lowe's price index)}.
\end{equation}

A critique of circularity and time reversibility (Pfouts)

Circularity is tantamount to the requirement that a certain matrix $P$ of index numbers has to be *singular*. $P$ is defined as follows (in the case of $T+1 = 4$ rows and columns, $t = 0, 1, ..., T$)
Fisher’s tests, however, tacitly assume \( P \) being singular. This can easily be seen since in the case of \( T = 2 \) we obtain:

\[
P = \begin{bmatrix}
P_{00} & P_{01} & P_{02} & P_{03} \\
P_{10} & P_{11} & P_{12} & P_{13} \\
P_{20} & P_{21} & P_{22} & P_{23} \\
P_{30} & P_{31} & P_{32} & P_{33}
\end{bmatrix}
\]

and the determinant \( |P| \) in fact vanishes. A consequence is that a single additional value, \( P_{23} \), is sufficient to calculate a fourth row and column; although we do not even have to know what index formula is being used.

\[
P_{c} = \begin{bmatrix}
1 & P_{01} & P_{02} \\
1/P_{01} & 1 & P_{12} \\
1/P_{01}P_{12}/P_{23} & 1/P_{12} & 1
\end{bmatrix}
\]

There are actually only two independent observations, \( P_{01} \) and \( P_{12} \) assembled in the \( 3 \times 3 \) matrix \( P \). It can easily be verified that time reversibility implies all second order principal minors of \( P \) being identically singular, hence determinants like

\[
\begin{vmatrix}
P_{00} & P_{01} \\
P_{10} & P_{11}
\end{vmatrix}
\]

or

\[
\begin{vmatrix}
P_{33} & P_{35} \\
P_{53} & P_{55}
\end{vmatrix}
\]

will all vanish.

### 2.3. The stochastic approach in price index theory

The underlying conception of the new stochastic approach (NSA) is to find regression equations in which index formulas are playing the part of regression coefficients. If applied to empirical data the error term in the regression is taken as an indication for the “reliability” or appropriateness of the index function. Goodness of fit is interpreted as caused by the dispersion in the (sample) price data and the index formula in question. The NSA allows to estimate standard errors and confidence intervals of various index formulas and thereby (ostensibly) a better understanding of index formulas.

#### Figure 2.3.1: The place of the stochastic approach in index theory

<table>
<thead>
<tr>
<th>atomistic(^1) approach</th>
<th>economic approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>dealing with observable variables only, and treating prices (( p )) and quantities (( q )) as being independent(^2)</td>
<td>treats the ( q )'s are depending on ( p )'s (as made explicit by the concept of “indifference curves” (preference orderings))</td>
</tr>
<tr>
<td>axiomatic approach</td>
<td>stochastic approach</td>
</tr>
<tr>
<td>aspects of interpretation are introduced via “axioms”</td>
<td>statistical criteria should decide on the choice among index formulas</td>
</tr>
<tr>
<td>old stochastic approach (OSA)</td>
<td>new stochastic approach (NSA)</td>
</tr>
<tr>
<td>index as a mean (unweighted) of the distribution of price relatives</td>
<td>index as a regression coefficient (unknown parameter) in a model to explain price variation</td>
</tr>
</tbody>
</table>

---

\(^1\) also called "formal" or "mechanic" (mechanistic) and the like

\(^2\) variations in \( q \) in response to variations in \( p \) are captured indirectly in the NSA since in some regression models expenditure shares are involved
Figure 2.3.2: Some "Budget share weighted average models" (BSW) models

General structure of the model \( y_{it} = \theta x_{it} + u_{it} \), where \( u_{it} \) is a function of \( \varepsilon_{it} \) and \( \varepsilon_{it} \) fulfills the standard assumptions concerning \( E(\varepsilon_{it}), V(\varepsilon_{it}), \) and \( C(\varepsilon_{it}, \varepsilon_{jt}) \).

<table>
<thead>
<tr>
<th>index formula (^1)\</th>
<th>y-variable ( p_{it} )</th>
<th>x-variable ( \sqrt{w_{it}} )</th>
<th>error term u ( \varepsilon_{it} )</th>
<th>((n-1)\hat{\sigma}_{\theta}^2) (^2)\</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laspeyre's ( P_{t0}^L )</td>
<td>( \sqrt{w_{i0}} )</td>
<td>( \sqrt{w_{i0}} )</td>
<td>( \varepsilon_{it} \sqrt{w_{i0}} )</td>
<td>( \sum w_{i0} \left(p_{it}^L - \hat{\theta}_t\right)^2 )</td>
</tr>
<tr>
<td>Paasche ( P_{t0}^P )</td>
<td>( \sqrt{w_{it}} )</td>
<td>( \sqrt{w_{it}} )</td>
<td>( \varepsilon_{it} \sqrt{w_{it}} )</td>
<td>( \sum w_{it} \left(p_{it}^P - \hat{\theta}_t\right)^2 )</td>
</tr>
<tr>
<td>ln(P(^T)) Törnquist</td>
<td>( Dp_{it} )</td>
<td>( \varepsilon_{it} \sqrt{w_{it}} )</td>
<td>( \sum w_{it} \left(p_{it}^T - \hat{\theta}_t\right)^2 )</td>
<td></td>
</tr>
<tr>
<td>ln(P(^J) Jevons</td>
<td>( Dp_{it} )</td>
<td>( \varepsilon_{it} )</td>
<td>( \frac{1}{n} \sum \left(Dp_{it} - \hat{\theta}_t\right)^2 )</td>
<td></td>
</tr>
</tbody>
</table>

1) parameter \( \hat{\theta}_t \) equals formula of ...; 2) \( \hat{\sigma}_{\theta}^2 \) denotes the estimated sampling variance of the regression coefficient \( \hat{\theta}_t \); 3) variance \( V(\varepsilon) = \sigma^2 \)

The centerpiece of the NSA is to regard empirical estimates as reflecting the suitability of an index formula. It is our view, however, that this relationship between an index formula on the one hand and the fit of a regression (when applied to data) on the other hand is a misconception.

(2.3.1) \( p_{it}^0 = \theta_t + \varepsilon_{it} \),

It is clear that Carli’s index (unweighted arithmetic mean of price relatives) will be the least squares (LS) estimator of \( \theta_t \) in the model of eq. 1, that is \( \hat{\theta}_t = \sum p_{it}^0 / n = P_{0t}^C \).

By the same token we get Jevons’ index (unweighted geometric mean of price relatives)

(2.3.2) \( Dp_{it}^1 = \ln(p_{it}^1) = \theta_t + \varepsilon_{it} \).

Example 2.3.1 The following data for \( n = 5 \) commodities are given

<table>
<thead>
<tr>
<th>( i )</th>
<th>( p_0 )</th>
<th>( p_t )</th>
<th>( q_0 )</th>
<th>( q_t )</th>
<th>( w_{i0} )</th>
<th>( w_{it} )</th>
<th>( \sqrt{w_{i0}} )</th>
<th>( \sqrt{w_{it}} )</th>
<th>( Dp_{it}^1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>15</td>
<td>12</td>
<td>20</td>
<td>35</td>
<td>0.200</td>
<td>0.1750</td>
<td>0.1392</td>
<td>0.1875</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>30</td>
<td>20</td>
<td>25</td>
<td>10</td>
<td>0.225</td>
<td>0.2083</td>
<td>0.1491</td>
<td>0.2167</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>32</td>
<td>15</td>
<td>10</td>
<td>0.100</td>
<td>0.2000</td>
<td>0.2545</td>
<td>0.1500</td>
<td>0.750</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>30</td>
<td>25</td>
<td>20</td>
<td>30</td>
<td>0.300</td>
<td>0.3125</td>
<td>0.3579</td>
<td>0.3063</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>12</td>
<td>20</td>
<td>17.5</td>
<td>15</td>
<td>0.175</td>
<td>0.1042</td>
<td>0.0994</td>
<td>0.1396</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>index ( \hat{\theta}_t )</th>
<th>((n-1)\hat{\sigma}_{\varepsilon}^2)</th>
<th>( \hat{\sigma}<em>{\theta}^2 = \sqrt{\hat{\sigma}</em>{\varepsilon}^2 / n} )</th>
<th>bounds of confidence interval</th>
<th>length of conf. interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laspeyre's ( P_{0t}^L ) = 1.04</td>
<td>0.0460</td>
<td>0.04795</td>
<td>0.906877; 1.17131</td>
<td>0.266599</td>
</tr>
<tr>
<td>Paasche ( P_{0t}^P ) = 0.9543</td>
<td>0.0459</td>
<td>0.04790</td>
<td>0.82128; 1.0827</td>
<td>0.26599</td>
</tr>
<tr>
<td>logarithm of Törnquist ( \ln(P_{0t}^T) ) = -0.00247</td>
<td>0.0448</td>
<td>0.04835</td>
<td>-0.13671; 0.13177</td>
<td>0.26862</td>
</tr>
<tr>
<td>retransformed ( P_{0t}^T ) = 0.998</td>
<td>0.87222; 1.14084</td>
<td>0.268609</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \ln(P_{0t}^N) ) = 0</td>
<td>0.0460</td>
<td>0.04835</td>
<td>-0.13390; 0.13390</td>
<td>0.268609</td>
</tr>
<tr>
<td>retransformed ( P_{0t}^N ) = 1</td>
<td>0.87467; 1.143283</td>
<td>0.268609</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 2.3.3: Main ideas of the new stochastic approach

<table>
<thead>
<tr>
<th>data</th>
<th>index functions</th>
<th>results of calculations</th>
</tr>
</thead>
<tbody>
<tr>
<td>n different price relatives</td>
<td>different index formulas $\Theta_1, \Theta_2 \ldots$ applied</td>
<td>different formulas will yield different results*)</td>
</tr>
</tbody>
</table>

- variability of price relatives gives rise to a disturbance term
- index function as regression coefficient $\Theta$ (explanation?)
- the standard error (S.E.) of the estimate of $\Theta$ (criterion for model-fit)

Main conclusions and rules of NSA

1. understand "index" by identifying it as a parameter in a regression equation
2. take standard error as an indication of the "reliability" or appropriateness of the index function $\Theta$ (prefer $\Theta_1$ to $\Theta_2$ when the S.E. of $\Theta_1$ is smaller than of $\Theta_2$)

*) It is the essence of the NSA method that the direction of this last arrow can be inverted in that the variability of the results permits an assessment of the index formulas.

2.4. Economic approach (the "true cost of living index", COLI)

The "true cost of living index" (COLI), or "constant utility index" (CU-index) is defined as the ratio of the minimum expenditures required to attain a particular indifference curve (the same utility level) under two price regimes (or price vectors $p_t$ and $p_0$), or: it is the amount of income necessary to leave somebody as well off as before the price change. The utility function (graphically represented by the well known "indifference curve") assigns a certain utility level (welfare) $U_0$ to infinitely many combinations of quantities $q_1$ and $q_2$ at base time $0$ as follows: $U_0 = U(q_{10}, q_{20})$, the function $U$ and it value $U_1$ at time 1 being analogously defined. Maximizing the utility $U$ under the constraint of balancing the "budget" (expenditure function $y_0$) that is

$$\max U(q_{10}, q_{20}) \text{ given } y_0 = p_{10}q_{10} + p_{20}q_{20}$$

has a unique solution (a tangential point of the budget line and the indifference curve) determining the quantities $q_{10}$ and $q_{20}$ and thereby the minimum expenditure $y_0 = y(p_{10}, p_{20}, U_0)$ such that

$$P_{0t}^{CU}(U_0) = \frac{y(p_1, U_0)}{y(p_0, U_0)} = \frac{C(t,0)}{C(0,0)}$$

is the "constant utility" (CU) – or "true cost of living" COLI-index. Accordingly the ratio of the minimum expenditures ($y$), or cost $C$ required to attain the same utility level $U_1$ is given by

$$P_{0t}^{CU}(U_t) = \frac{y(p_1, U_t)}{y(p_0, U_t)} = \frac{C(t,1)}{C(0,1)}.$$  

The CU-index depends not only on price vectors but also on the utility level in question, that means that the two indices in eqs. 1 and 2 are not necessarily equal unless in the case of homothetic indifference curves, or (equivalently): a linear homogenous (in the quantities) utility function. Note that

The Laspeyres price index compares expenditures at different price regimes referring to the same quantities whilst the CU-index compares expenditures referring to the same utility.

$$P_{0t}^{CU}(U_0) \leq P_{0t}^L \text{ (upper bound) and lower bound}$$

$$P_{0t}^P \leq P_{0t}^{CU}(U_t)$$

provided that indices $P^L$ and $P^P$ refer to a single utility maximizing household "possessing" indifference curve $U_0$ and $U_t$ respectively.
A system of indifference curves (IC) is called *homothetic* (or *linear homogeneous*) if 
\[ U(\lambda q) = \lambda U(q), \lambda \neq 0 \]; then each IC is a uniform enlargement, or contraction of each other, 
and the inequalities 3a/3b can be combined into one single inequality 
\[ P_{t1}^P \leq P_{t1}^{UC}(U) \leq P_{t1}^L. \]

Critique of the "utility" reasoning in the COLI-approach:

1. The distinction between inflation and welfare measurement becomes blurred, questionable 
imputations of gains or losses in utility are instigated, and 2. the notion of "good" becomes 
boundless, and finally 3. we move away from statistics of observable phenomena to specu-
lations about levels of utility or a "fair" amount of income necessary for a "compensation".

### 2.5. Chain indices and Divisia's approach (general introduction)

| a) Necessary terminological distinctions | c) Divisia index, its relation to chain indices |
| b) Weights in the chain approach | d) Discrete time approximations and weights |

#### a) Some necessary terminological distinctions

A distinction between "chain" and "fixed based" or "fixed weighted" index is misleading and 
should be avoided. It is better to distinguish chain- and direct indices.

**Figure 2.5.1:** Terminological distinctions referring to chain indices

- **a) Types of comparison** between 0 and t
  - Direct index approach: using data of 0 and t only
  - Chain index approach: index defined as a product of links

- **b) The two elements to define a chain index**
  - Constant element: the "chain" \( P_{t1} \), an index to compare 0 and t, gained by multiplication of links
  - Variable element: definition of a "link" \( P_{t1}^C \) (index to compare two adjacent periods \( P_{t1}^C = P_{t1,t2} \))
  - Examples: Laspeyres-, Paasche-, Fisher-type

A chain index essentially is a specific type of temporal aggregation and description of a time 
series rather than a comparison of two states taken in isolation, it provides a measure of the 
cumulated effect of successive steps (and the shape of the path) from 0 to 1, 1 to 2, ..., t-1 to t.

#### Three sources of variation

1. the difference in prices in t as compared with 0,
2. the change weights (quantities) have undergone (in response to a change in prices) in 
   comparing 0 and t, and
3. prices and quantities in the intermediate points in time, that is in 1, 2,.., t-1.

---

4 A fourth source is the ever changing “domain of definition” of the index function, which is the often praised 
 ease with which the selection of goods can be changed from one period to another (not only in the case of re-
basing).
The two "elements" of the definition of chain price\(^5\) indices should be kept distinct:

1. A constant element is the "chain" \(P_{0t}\), which always is a product of "links" \(P^L_t\), each of which being a direct index comparing \(t\) with the preceding period \(t-1\)

\[
(2.5.1) \quad P_{0t} = P^C_1 P^C_2 \ldots P^C_t = \prod_{i=1}^{t} P^C_i, \quad \text{and}
\]

2. In defining the link \(P^C_t = P_{t-1,t}\) there are numerous solutions we might think of, giving rise to Laspeyres-, Paasche- Fisher- and other chain index numbers (depending on the type of link \(P^L_{1t}, P^C_t, P^L_{t-1}\) etc. that are multiplied ["chainlinked"] to get the chain \(P_{0t}\)).

The Laspeyres link, as an example is defined as follows

\[
(2.5.2) \quad P^L_{1t} = \frac{\sum p_i q_{t-1}}{\sum p_i q_t}, \quad \text{such that} \quad P^L_{0t} = P^L_{1t} \ldots \ldots \ldots \quad \text{is the Laspeyres chain.}
\]

Since a link always compares the reference period \(t\) with the preceding base period \(t-1\) there is no need for two subscripts. It is sufficient to use only one subscript, \(t\). The Paasche link obviously is \(\Sigma p_i q_t / \Sigma p_i q_{t-1}\).

Due to the multiplication the chain \(P_{0t}\) is in general a function of the price and quantity vectors \(p_0, q_0, p_1, q_1, p_2, q_2, \ldots, p_{t-1}, q_{t-1}, p_t, q_t\), and not only of the first and last pair of vectors.

Note that the existence of a product representation of an index as such is not sufficient to characterize a chain index

\[
(2.5.3) \quad P^L_{03} = \left( \sum_{p_0} \frac{p_1}{p_0} \sum_{p_0} q_0 \right) \left( \sum_{p_1} \frac{p_2}{p_1} \sum_{p_1} q_1 \right) \left( \sum_{p_2} \frac{p_3}{p_2} \sum_{p_2} q_2 \right)
\]

\[
(2.5.4) \quad P^L_{01} = \left( \sum_{p_0} \frac{p_1}{p_0} \sum_{p_0} q_0 \right) \left( \sum_{p_1} \frac{p_2}{p_1} \sum_{p_1} q_0 \right) \left( \sum_{p_2} \frac{p_3}{p_2} \sum_{p_2} q_0 \right) = \sum_{p_3} \frac{p_0}{p_0} \sum_{p_0} q_0.
\]

Note also that the factors on the right hand side (RHS) of eq. 4 are not the "ordinary" indices, \(P^L_{01}, P^L_{12}\) and \(P^L_{03}\) (since \(P^L\) is not transitive), but a sequence of rebased Laspeyres indices

\[
P_{01} = \frac{p_{01}}{p_{00}} = \sum_{p_{01}} \frac{p_{00}}{p_{01}} q_0, \quad P_{12} = \frac{p_{22}}{p_{01}} = \sum_{p_{22}} \frac{p_{01}}{p_{22}} q_0, \quad \text{and} \quad P_{23} = \frac{p_{03}}{p_{22}} = \sum_{p_{03}} \frac{p_{22}}{p_{03}} q_0.
\]

\(P^L_{03}\) will in general differ from \(P^L_{03}\) which is known as drift of the chain index\(^7\). Though by definition the following holds

\[
(2.5.5) \quad P_{0t} = P^L_{0k} P^C_{kt}
\]

The idea of the chain test (or "chainability" or "transitivity") is that the result (\(P_{0t}\)) should be the same for any \(k\), irrespective of how the interval \((0, t)\) is partitioned into sub-intervals. But in general this is not true in the case of chain indices. Not only is \(P^L_{06}\) different from \(P^L_{06}\), the case of 1-period-links, \(P^L_{06} = P_{01} P_{12} \ldots P_{56}\) will in general not yield the same result as for example the chaining of 2-period-links \(P_{02} P_{24} P_{46}\) (see chapter 7 for more details).

---

\(^5\) The definition applies mutatis mutandis also to quantity indices.

\(^6\) The link thus is the variable element of the definition of a chain index.

\(^7\) The term "drift" does not mean that the incorrect chain index is drifting away from the correct direct index. We may of course as well think of the direct index drifting away from the (correct) chain index.
The name "chain index" is misleading: multiplication ("chaining") may give rise to the impression chainability were met. But this is not true. Moreover: Multiplication is not a unique, defining feature of chain indices. Nor are there any desirable properties of chain indices to be concluded from multiplication as such.\footnote{8}

\[
(2.5.3a) \quad \bar{p}_{03}^{LC} = \frac{\sum_i p_{i1}q_{i0} \sum_k p_{2k}q_{ik} \sum_m p_{3m}q_{2m}}{\sum_i p_{0i}q_{i0} \sum_k p_{1k}q_{ik} \sum_m p_{2m}q_{2m}} \quad \text{(Changes in the domains of definition),}
\]

b) Weights in the chain approach

The typology of the following \textbf{fig. 2.5.2} is obscuring and we prefer the typology of \textbf{fig. 2.5.3}.

\textbf{Figure 2.5.2:} Traditional classification of weighting schemes \pagefootnote{\(a\)}

\begin{center}
\begin{tabular}{|c|c|}
\hline
\textbf{Types of indices} & \\
\hline
\textbf{fixed based} & \textbf{variably based} \\
\hline
period of weights held constant & as time \( t \) goes on \\
\hline
base weighted & fixed weighted \\
Laspeyres form & weight base other than 0 or \( t \) \\
\hline
chain based & other designs \\
the previous year as (weight) base \pagefootnote{\( b \)} & e.g. Paasche index or crossed formulas \\
\hline
\end{tabular}
\end{center}

\( a \) Other possibilities exist. The weight base can be a period other than 0 or \( t \) in the interval \((0,t)\) or it can be periodic (e.g. each starting point of a business cycle).

\( b \) this, however, applies only to the links

\textbf{Figure 2.5.3:} Alternative principles to define weights

\begin{center}
\begin{tabular}{|c|c|}
\hline
\textbf{Weights in an index} \( P_0 \), comparing 0 and \( t \) * & \\
\hline
\textbf{single} weights relating to one period only & \textbf{multiple} weights, relating to more than one period \\
\hline
average weights, mostly relating to two periods, 0 and \( t \) & chain index: cumulative weights, relating to all preceding periods \\
\hline
\end{tabular}
\end{center}

\* This scheme compares \textbf{direct} indices with \textbf{chains}, \( \text{not direct indices with links} \) (as \textbf{fig. 2.5.2} does). A direct index will in general be single-weighted, like \( P_L \) or \( P_P \), or average weighted, like \( P_{ML} \), \( P_T \) or \( P_W \), whereas a chain index \textit{always} has cumulative weights.

c) Divisia index and its relation to chain indices

It is assumed that two functions, \( p_i(t) \) and \( q_i(t) \) exist \textit{for each commodity} \((i = 1, \ldots, n)\) \textit{at any point in time} (a continuous variable). Let \( P(\tau) \) denote the (unknown, by contrast to \( p_i(t) \)) price \textit{level} function (absolute aggregative level) varying continuously over time and let \( Q(\tau) \) denote the quantity level defined analogously. It is a mere matter of definition that a value function \( V(\tau) \) exists as follows

\[
(2.5.6) \quad V(\tau) = \sum_{i=1}^{n} p_i(\tau)q_i(\tau),
\]

\footnote{8}{On the contrary: to study figures resulting from a complicated mix of influences makes, in general less sense.}
\( V(\tau) = P(\tau) Q(\tau) \).

Unlike the functions \( p_i(\tau) \) and \( q_i(\tau) \) the levels \( P(\tau) \) and \( Q(\tau) \) are unobservable. They will lead eventually to a "price index" and "quantity index" respectively. Eq. 7 only defines the "levels" \( P(\tau) \) and \( Q(\tau) \) implicitly by stipulating a relation between \( V(\tau) \), \( P(\tau) \) and \( Q(\tau) \).

Consider differential changes of \( V(\tau) \) according to eq. 6

\[ dV(\tau) = \sum_i q_i(\tau) dp_i(\tau) + \sum_i p_i(\tau) dq_i(\tau). \]

Dividing both sides by \( V(\tau) \) as given in eq. 6 leads to

\[ \frac{dV(\tau)}{V(\tau)} = \frac{\sum q_i(\tau) dp_i(\tau) / d\tau}{\sum p_i(\tau) q_i(\tau)} + \frac{\sum p_i(\tau) dq_i(\tau) / d\tau}{\sum p_i(\tau) q_i(\tau)}. \]

The (continuous time) growth rate of \( V(\tau) \) is the sum of the growth rate of the price level \( P(\tau) \) and the quantity level \( Q(\tau) \) respectively. Taking the growth rate of \( P(\tau) \) for example we see that it is a weighted average of the growth rates of the prices \( p_i(\tau) \) of the individual commodities \( i = 1, \ldots, n \)

\[ \frac{dP(\tau)}{P(\tau)} = \frac{\sum q_i(\tau) dp_i(\tau)}{\sum p_i(\tau) q_i(\tau)} = \sum w_i(\tau) \frac{dp_i(\tau)}{p_i(\tau)} = \sum w_i(\tau) \frac{d\ln p_i(\tau)}{d\tau}, \]

where weights \( w_i(\tau) = p_i(\tau) q_i(\tau) / \sum p_i(\tau) q_i(\tau) \) are expenditure shares at point \( \tau \) (and hence changing with time) and summation takes place over \( n \) commodities. In the same manner the growth rate of \( Q(\tau) \) is a weighted arithmetic mean of growth rates of \( n \) functions \( q_i(\tau) \)

\[ \frac{dQ(\tau)}{Q(\tau)} = \frac{\sum q_i(\tau) dq_i(\tau)}{\sum p_i(\tau) q_i(\tau)} = \sum w_i(\tau) \frac{d\ln q_i(\tau)}{d\tau}. \]

Justification to identify \( P(\tau) \) as price level at time \( \tau \) (and \( P(t)/P(0) \) as price index):

Assume quantities in eq. 9 don't change such that \( dq_i(\tau) = 0 \) for all \( i \) then
- the change of the quantity level (that is of \( Q(\tau) \)) according to eq. 11 should also be zero, or in other words
- the change of volume (in eq. 9) should equal the change of prices.

This applies mutatis mutandis for the assumption of no change of prices \( dp_i(\tau) = 0 \).

It is this consideration that allows to separate the two differentials (in prices and in quantities) and to identify them as growth rate of price and quantity level respectively.

Another way of looking at Divisia's method is indicated in eq. 10 and 11: a (continuous time) growth rate of the (absolute) price level \( P(\tau) \), or quantity level \( Q(\tau) \) is constructed as a weighted sum of \( n \) growth rates; weights \( w_i(\tau) \) being shares in total value and varying continuously with time. To get a price index the differential equation (eq. 10) is to solve ("integrate") for \( P \).
\[(2.5.12) \quad P(t) = P(0) \exp \left( \int_0^t \sum_i q_i(\tau) \frac{dp_i(\tau)}{dt} \, d\tau \right) = P(0) \exp \left( \int_0^t \sum_i w_i(\tau) \frac{d\ln p_i(\tau)}{d\tau} \, d\tau \right), \]

and thus the price index is given by
\[(2.5.12a) \quad P_{0t}^{Div} = \frac{P(t)}{P(0)}. \]

Correspondingly Divisia's quantity index is
\[(2.5.13) \quad Q_{0t}^{Div} = \frac{Q(t)}{Q(0)} = \exp \left( \frac{1}{\sum_i p_i(\tau) q_i(\tau)} \int_0^t \sum_i p_i(\tau) dq_i(\tau) \right). \]

The name "integral index" stems from the fact that the pair of Divisia indices is derived by solving (integrating) differential equations. The problem, however, is that the integration suffers from lack of path invariance: the solutions (integral functions) of the "line integrals" in eq. 13 and 14 depend on the path connecting 0 and t. By contrast the integration
\[(2.5.14) \quad V_{0t} = \frac{V(t)}{V(0)} = \exp \left( \int_0^t \frac{dV(\tau)}{V(\tau)} \, d\tau \right) = \exp \left( \int_0^t \frac{dV(\tau)}{V(\tau)} \right) = \exp[f(0,t)] \]

depends on the endpoints 0 and t only, not on the (shape of the) path connecting them. Thus
\[(2.5.14a) \quad V_{0t} = V_{0k} V_{kt} = \exp \left( \int_0^k \frac{dV}{V} + \int_k^t \frac{dV}{V} \right) = \exp \left( \int_0^t \frac{dV}{V} \right) \text{ or } f(0,t) = f(0,k) + f(k,s). \]

By contrast to \(f(\tau)\) the corresponding functions in eq. 12 and 13 are not path invariant. 9

Not only chain indices but also certain direct indices can be derived
- under specific assumptions concerning the functions \(p_i(\tau)\) and \(q_i(\tau)\), or
- from various types of discrete time approximations to the continuous time Divisia index.

Consider in \(\tau\) the same (or for all \(i\) proportional) quantities as compared with 0, or assume a (most unlikely) path of quantities such that \(q_i(\tau) = \lambda q_i(0) = \lambda q_0\). This gives
\[(2.5.15) \quad \frac{dP(\tau)}{P(\tau)} = \frac{\sum q(\tau) dp(\tau)}{\sum q(\tau) p(\tau)} = \frac{\sum \lambda q_0 dp(\tau)}{\sum \lambda q_0 p(\tau)} = \frac{d \sum q_0 p(\tau)}{\sum q_0 p(\tau)}, \]

where the subscript \(i\) denoting the commodity is deleted for convenience. Integration of the price differential \(\int_0^t \frac{dP(\tau)}{P(\tau)} \, d\tau\) subject to constant or proportional quantities10 leads to
\[(2.5.16) \quad \ln P(t) = \ln \left[ \sum \lambda q_0 p_1 \right] + C \quad \text{where} \quad p_1 = p(\tau = t) \]

with an arbitrary constant \(C\), the value of which can be determined by assuming prices \(p_0\) to enter \(P(0)\) the price level in base period 0 as do prices \(p_t\) with \(P(t)\). This means
\[(2.5.16a) \quad \ln \frac{P(t)}{P(0)} = \ln P(t) - \ln P(0) = \ln \frac{\sum \lambda q_0 p_0}{\sum p_0 q_0}. \]

---

9 Divisia himself was already aware of this drawback and he proposed chain indices as a discrete approximation.

10 Or: in which the individual price changes are weighted with constant base period weights.
such that we finally arrive at the Laspeyres index $P(t)/P(0) = \sum p_t q_0/\sum p_0 q_0$. Of course $P^L$ is path invariant as opposed to $P^{Div}$. It can be shown that $P^p$ and $P^f$ also might be regarded as special cases of the Divisia index.

d) Discrete time approximations and weights in Divisia’s approach

Substituting forward differences $\Delta V_t = V_{t+1} - V_t = \sum p_{t+1} q_{t+1} - \sum p_t q_t$ for the differential $dV$ (and correspondingly $\Delta p_{it}$ and $\Delta q_{it}$ for $dp$ and $dq$) leads to

$$\Delta V_t = \sum q_{it} \Delta p_{it} + \sum q_{it} \Delta q_{it} + \sum \Delta p_{it} \Delta q_{it},$$

an equation equivalent to eq. 8 however with a mixed element $\sum \Delta p_{it} \Delta q_{it}$. It is reasonable therefore to define $P_{t}$ and $Q_{t}$ in such a way that

$$\Delta \frac{P_t}{P_t} = \sum q_{it} \Delta p_{it} \text{ and } \frac{Q_t}{Q_t} = \sum q_{it} \Delta q_{it} \text{. Using eq. 17 this gives}$$

$$\Delta \frac{V_t}{V_t} = \Delta \frac{P_t}{P_t} + \Delta \frac{Q_t}{Q_t} + \sum \Delta p_{it} \Delta q_{it},$$

or

$$\Delta \frac{V_t}{V_t} = \left(\frac{P_{t+1}}{P_t} - 1\right) + \left(\frac{Q_{t+1}}{Q_t} - 1\right) + R_t,$$

The residual term $R_t = \sum \Delta p_{it} \Delta q_{it}$ will tend to zero and can be neglected. Thus

$$\frac{P_{t+1}}{P_t} = \sum q_{it} \frac{P_{t+1}}{P_t} = \frac{P_t^{LC}},$$

which is the Laspeyres link, and the corresponding index (comparing period $t$ with 0) then is

$$\frac{P_{t+1}}{P_0} = \frac{P_{t+1}}{P_0} \frac{P_{t+1}}{P_t} = \frac{P_t^{LC} P_{t+1}^{LC}}{P_t},$$

and analogously

$$\frac{Q_{t+1}}{Q_0} = \frac{Q_{t+1}}{Q_{t+1}} = \frac{Q_{t+1}^{LC}}{Q_{t+1}},$$

In a similar manner we may derive Paasche chain indices $\bar{P}_{0t}^{PC}$ (and $\bar{Q}_{0t}^{PC}$ ) by using backward differences $\Delta^* P_t = P_t - P_{t-1}$ and $\Delta^* p_{it} = p_{it} - p_{i,t-1}$ respectively

$$\Delta \frac{P_t}{P_{t-1}} = 1 - \sum q_{it} P_{t-1} = 1 - \frac{P_t}{P_{t-1}} = 1 - \left(\frac{P_t}{P_{t-1}}\right)^{-1} = 1 - \left(P_{t-1}^{PC}\right)^{-1}.$$

This consideration does not, however, support the often heard statement, that the correctness of the chain approach has been proved by Divisia’s formula:

"... the smaller we make the unit of time or space within which production or consumption takes place, the less actual production or consumption there will be to observe, and comparisons between these tiny units will become meaningless". (Diewert and Nakamura 1993, p. 3).

"The problem with this approach is that economic data are almost never available as continuous time variables ... Hence for empirical purpose it is necessary to approximate the continuous time
Divisia price and quantity indexes by discrete time data. Since there are many ways of performing these approximations, the Divisia approach does not seem to lead to a definite result". (p. 23)\textsuperscript{11}

2.6. Additive models, Stuvel’s and Banerjee’s index formulas

This is an additive approach to index numbers (fig. 2.6.1). The additive analysis is even on the microlevel not uniquely determined. Hence the following two equations with relatives \( p_i^* = \frac{p_{it}}{p_{i0}} \) and \( q_i^* = \frac{q_{it}}{q_{i0}} \) will hold

\[(a1) \quad v_{it} - v_{i0} = (v_{it} - v_{i0}p_i^*) + (v_{i0}p_i^* - v_{i0}) = v_{i0}p_i^*(q_i^* - 1) + v_{i0}(p_i^* - 1)\]

\[(a2) \quad v_{it} - v_{i0} = (v_{it} - v_{i0}q_i^*) + (v_{i0}q_i^* - v_{i0}) = v_{i0}q_i^*(p_i^* - 1) + v_{i0}(q_i^* - 1)\]

Summation over all \( n \) commodities and division by \( v_{0t} = \sum v_{i0} = \sum p_{0t}q_{0t} \) yields (omitting the subscripts 0 and \( t \) for convenience of presentation) \( V = V_{0t} \)

\[(A1^*) \quad V - 1 = (V - P^L) + (P^L - 1) = P^L(Q^P - 1) + (P^L - 1)\]

\[(A2^*) \quad V - 1 = (V - Q^L) + (Q^L - 1) = Q^L(P^Q - 1) + (Q^L - 1)\]

Note that all equations presented so far are nothing but simple identities.

Figure 2.6.1: Stuvel’s approach

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{a) Types of analysis} & \textbf{microlevel} & \textbf{macrolevel} \\
\hline
\text{multiplicative approach} & \( \frac{v_{it}}{v_{i0}} = \frac{p_{it}}{p_{i0}} \frac{q_{it}}{q_{i0}} \) & \( V_{0t} = \frac{\sum v_{it}}{\sum v_{i0}} = P_{0t} Q_{0t} \) \\
\hline
\text{additive approach} & \( \Delta v_i = v_{it} - v_{i0} = A_i + B_i \) & \( \Delta V = \sum v_{it} + \sum v_{i0} = \sum A_i + \sum B_i = A + B \) \\
\hline
\end{tabular}
\end{table}

\begin{tabular}{|c|c|}
\hline
\textbf{terms} \( A_i \) and \( B_i \) (or \( A \) and \( B \) on the macrolevel) are supposed to measure price and quantity component \\
\hline
\textbf{comments} & Decomposition is uniquely determined in the single commodity case (microlevel) by equation \( (m) \), but on the macrolevel (in eq. \( M \)) only \( V \) is determined uniquely, \( P \) and \( Q \) are not. The values of \( P \) and \( Q \) will depend on what indices are chosen for their measurement. \\
\hline
\text{additive approach} & The additive analysis is even for single commodities (microlevel) not uniquely determined. Two eq. possible \( (a1 \text{ and } a2) \) to measure price and quantity component \\
\hline
\end{tabular}

The two decompositions (equations) in additive analysis

\begin{tabular}{|c|c|}
\hline
\textbf{microlevel} & \textbf{macrolevel} \\
\hline
\( (a1) \quad \Delta v_i = v_{i0}p_i^*(q_i^* - 1) + v_{i0}(p_i^* - 1) \) & \( (A1) \quad \Delta V = V_0(P - 1) + V_0(P - 1) \) \\
\hline
\( (a2) \quad \Delta v_i = v_{i0}q_i^*(p_i^* - 1) + v_{i0}(q_i^* - 1) \) & \( (A2) \quad \Delta V = V_0(Q - 1) + V_0(Q - 1) \) \\
\hline
\end{tabular}

\( p_i^* \) and \( q_i^* \) denote the price relative and quantity relative respectively.

\textsuperscript{11} More important still, since the approximations "can differ considerably (in amount), the Divisia approach does not lead to a practical resolution of the price measurement problem" (p. 43). Emphasis and text in brackets added.
To define the two components of the (absolute) value change, \( A \) and \( B \) a decision has to be made as to which equation should be used, \((A1^*)\) or \((A2^*)\). Since there is no indication why one of the two equations should be preferred to the other Stuvel calculated the average of both equations and arrived at two equations

\[
(*) \quad A + B = V_0(PQ - 1) \quad \text{and} \\
(**) \quad A - B = V_0(Q - P)
\]

Furthermore since \( V_t - V_0 = A + B \) (because of eq. A) we get \( V_{0t} = (A + B)/V_0 + 1 = PQ \) which simply states that Stuvel's pair of index numbers will meet the factor reversal test. Substituting \( P \) by \( V_{0t}/Q \) we get

\[
A - B = V_0 \left( Q - \frac{V_{0t}}{Q} \right) = V_0Q - \frac{V_t}{Q}
\]

and after some algebra we get the quadratic equation \( Q^2 + \frac{B-A}{V_0}Q - V_{0t} = 0, \)

and in a similar manner upon inserting \( V_{0t}/P \) for \( Q \) the following quadratic equation\(^{12}\) in \( P \)

\[
(2.6.1) \quad P^2 + \frac{A-B}{V_0}P - V_{0t} = 0.
\]

One of the two roots of this equation (taking into account \((A-B)/V_0 = Q^L - Q^P\)) is given by

\[
(2.6.2) \quad P = \frac{B-A}{2V_0} + \sqrt{\left( \frac{B-A}{2V_0} \right)^2 + V_{0t}} = \frac{P^L_{0t} - Q^L_{0t}}{2} + \sqrt{\frac{P^L_{0t} - Q^L_{0t}}{2}^2 + V_{0t}} = p^*_{0t}.
\]

\[
(2.6.3) \quad Q = \frac{A-B}{2V_0} + \sqrt{\left( \frac{A-B}{2V_0} \right)^2 + V_{0t}} = \frac{Q^L_{0t} - P^L_{0t}}{2} + \sqrt{\frac{Q^L_{0t} - P^L_{0t}}{2}^2 + V_{0t}} = q^*_{0t}.
\]

### Properties of Stuvel's index formulas \( P^*_{0t} \) and \( Q^*_{0t} \)

<table>
<thead>
<tr>
<th>axioms, &quot;tests&quot; and other criteria</th>
<th>satisfied</th>
<th>violated; further remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>all fundamental criteria</strong>, like dimensionality, com-mensurability, (strict) monotonicity, proportionality, factor reversal, time reversal test, consistency in aggregation and equality test satisfied</td>
<td>linear homogeneity (^1) <strong>not met</strong>, no interpretation as means of price relatives and in terms of costs of a budget (^3)</td>
<td></td>
</tr>
</tbody>
</table>

1) this is true also for the generalized Stuvel index except in case of \( P^L_{0t}, Q^P_{0t} \) and \( P^P_{0t}, Q^L_{0t} \).
2) that means: Stuvels indices are "ideal" index functions.
3) the Stuvel indices are difficult to interpret economically.

It is easy to see why linear homogeneity is violated

\[
P^*_{0t} (\lambda) = \frac{\lambda P^L_{0t} - Q^L_{0t}}{2} + \sqrt{\left( \frac{\lambda P^L_{0t} - Q^L_{0t}}{2} \right)^2 + \lambda V_{0t}} = \lambda P^*_{0t} \quad \text{because} \quad \frac{\lambda P^L_{0t} - Q^L_{0t}}{2} = \lambda \frac{P^L_{0t} - Q^L_{0t}}{2}.
\]

\(^{12}\) the non-negative solution of which renders Stuvel's price index.
Alternative ways of deriving Stuvel's indices and a generalization of Stuvel's formulas

\[
\begin{align*}
(A1^{**}) & \quad w(V - 1) = w(V - P^L) + w(P^L - 1) \\
(A2^{**}) & \quad (1-w) (V - 1) = (1-w)(V - Q^L) + (1-w)(Q^L - 1)
\end{align*}
\]

Special case \(P^{ST}/Q^{ST}\) is simply \(w = \frac{1}{2}\).

This idea is giving rise to a generalization of Stuvel's indices

Figure 2.6.2: An alternative way of deriving and interpreting \(P_{0t}^{ST}\) and \(Q_{0t}^{ST}\)

**first** condition: Find a pair of indices \(P\) (price index) and \(Q\) (quantity index) such that they pass factor re-
versal test \(V_{0t} = P_{0t}Q_{0t}\) (equation 2.6.5) and

<table>
<thead>
<tr>
<th>second condition (special)</th>
<th>second condition (general)</th>
</tr>
</thead>
<tbody>
<tr>
<td>give both types of additive decomposition ((A1, A2)) the same weight (\frac{1}{2}), or equivalently: (P - P^L_{0t} = Q - Q^L_{0t}) (P) should be equally away from (P^L_{0t}) as (Q) is away from (Q^L_{0t})</td>
<td>give additive decomposition ((A1, A2)) weights (w) and (1-w) respectively, equivalently: (w(P - P^L_{0t}) = (1-w)(Q - Q^L_{0t})) ((0 \leq w \leq 1)) (P^{ST}<em>{0t}) and (Q^{ST}</em>{0t}) is the special case of (w = \frac{1}{2}) → Generalized Stuvel indices</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
(2.6.8) \quad P^{ST}_{0t}(w) &= \frac{P^L_{0t} - \frac{1-w}{2}Q^L_{0t}}{w} + \sqrt{\left(\frac{P^L_{0t} - \frac{1-w}{2}Q^L_{0t}}{w}\right)^2 + \frac{1-w}{w} V_{0t}}, \text{ and} \\
(2.6.9) \quad Q^{ST}_{0t}(w) &= \frac{Q^L_{0t} - \frac{1-w}{2}P^L_{0t}}{1-w} + \sqrt{\left(\frac{Q^L_{0t} - \frac{1-w}{2}P^L_{0t}}{1-w}\right)^2 + \frac{w}{1-w} V_{0t}}.
\end{align*}
\]

Banerjee's factorial approach in index theory

\[
(2.6.10a) \quad P_{0t} = \frac{P^*}{P_0} \quad \text{and} \quad Q_{0t} = \frac{Q^*}{Q_0}. \quad \text{The approach leads to the following system of equations (notation)}
\]

<table>
<thead>
<tr>
<th>price 0</th>
<th>price t</th>
</tr>
</thead>
<tbody>
<tr>
<td>quantity 0</td>
<td>(Y_{00} = \sum p_0 q_0 = P^<em>_0 Q_0^</em>)</td>
</tr>
<tr>
<td>quantity t</td>
<td>(Y_{0t} = \sum p_0 q_t = P^<em>_0 Q^</em>_t)</td>
</tr>
</tbody>
</table>
Figure 2.6.4: Banerjee’s factorial approach (The system of equations)

<table>
<thead>
<tr>
<th>equation</th>
<th>derived from condition for</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ((P_{0t} + 1)(Q_{0t} + 1) = V_{0t} + P_{0t}^L + Q_{0t}^L + 1)</td>
<td>grand mean ((\mu))</td>
</tr>
<tr>
<td>(b) ((P_{0t} - 1)(Q_{0t} + 1) = V_{0t} + P_{0t}^L - Q_{0t}^L - 1)</td>
<td>factor price ((\alpha))</td>
</tr>
<tr>
<td>(c) ((P_{0t} + 1)(Q_{0t} - 1) = V_{0t} - P_{0t}^L + Q_{0t}^L - 1)</td>
<td>factor quantity ((\beta))</td>
</tr>
<tr>
<td>(d) ((P_{0t} - 1)(Q_{0t} - 1) = V_{0t} - P_{0t}^L - Q_{0t}^L + 1)</td>
<td>interaction ((\gamma))</td>
</tr>
</tbody>
</table>

Solution of the equations

<table>
<thead>
<tr>
<th>Combination of equations</th>
<th>Price index = (P_t^<em>/P_0^</em>)</th>
<th>Quantity index = (Q_t^<em>/Q_0^</em>)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a and b</td>
<td>index of Marshall Edgeworth</td>
<td>index of Laspeyres (Q_{0t}^L)</td>
</tr>
<tr>
<td>a and c</td>
<td>index of Laspeyres (P_{0t}^L)</td>
<td>index of Marshall Edgeworth</td>
</tr>
<tr>
<td>a and d</td>
<td>no meaningful result</td>
<td></td>
</tr>
<tr>
<td>b and c</td>
<td>index of Stuvel (P_{0t}^{ST})</td>
<td>index of Stuvel (Q_{0t}^{ST})</td>
</tr>
<tr>
<td>b and d</td>
<td>index of Laspeyres (P_{0t}^L)</td>
<td>new formula (Q_{0t}^{BA1} = Q_{0t}^L P_{0t}^P - 1)</td>
</tr>
<tr>
<td>c and d</td>
<td>new formula (P_{0t}^{BA1} = P_{0t}^L Q_{0t}^P - 1)</td>
<td>index of Laspeyres (Q_{0t}^L)</td>
</tr>
</tbody>
</table>

Hence formulas of Stuvel, Laspeyres and Marshall-Edgeworth, as well as two new formulas appear as special cases of the factorial approach.

Another pair of indices derived by Banerjee from his "economic Theory" approach

<table>
<thead>
<tr>
<th>index of</th>
<th>figure 2.6.4</th>
<th>economic theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>prices</td>
<td>(P_{0t}^{BA1} = P_{0t}^L Q_{0t}^P - 1)</td>
<td>(P_{0t}^{BA2} = P_{0t}^P (P_{0t}^L + 1))</td>
</tr>
<tr>
<td>quantities</td>
<td>(Q_{0t}^{BA1} = Q_{0t}^L P_{0t}^P - 1)</td>
<td>(Q_{0t}^{BA2} = Q_{0t}^P (Q_{0t}^L + 1))</td>
</tr>
</tbody>
</table>

It can be shown that both Banerjee-2 indices are bounded by the respective Laspeyres- and Paasche-indices \(P^L, Q^L\) and \(P^P, Q^P\). Given that \(P_{0t}^L = (P_{0t}^P + \Delta) > P_{0t}^P, \Delta > 0\) we obtain

\[ (P_{0t}^P + 1)P_{0t}^{BA2} = P_{0t}^P (P_{0t}^P + 1 + \Delta) \]

and thus \(P_{0t}^{BA2} = P_{0t}^P + \frac{\Delta P_{0t}^P}{P_{0t}^P + 1} > P_{0t}^P\) and in just the same manner we get \(P_{0t}^{BA2} < P_{0t}^L\) in this situation \((P_{0t}^P < P_{0t}^L)\), and \(P_{0t}^L < P_{0t}^{BA2} < P_{0t}^P\) in the case \(P_{0t}^L < P_{0t}^P\).

The Banerjee-1 indices are not bounded by the respective Laspeyres- and Paasche-indices, and these indices can even yield completely absurd results: whenever a Paasche quantity index displays decreasing quantities \((Q^P < 1)\) and the Laspeyres quantity index increasing quantities \((Q^L > 1)\) then \(P_{0t}^{BA1}\) will be negative (!). This applies also mutatis mutandis to \(P_{0t}^{BA1}\) with reference to \(P^P\) and \(P^L\).