Chapter 3 Axioms and more index formulas

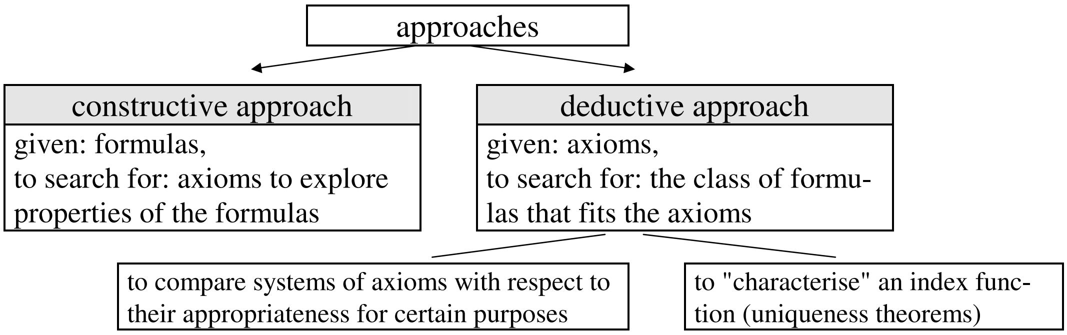
3.1. The axiomatic approach, some theorems and fundamental axioms

What is an axiom?

(3.1.1) $\varphi(y,x) = [\varphi(x,y)]^{-1}$ (a functional equation)

Example: the function $y = \sqrt{x}$ would not fulfil this since then $x \neq 1/\sqrt{y}$ but rather $x = y^2$.

Figure 3.1.2: Types of axiomatic approaches



A system of axioms A_1, A_2, … has to be **consistent** and **independent**: How to prove

consistency	independence
inconsistency	dependence

(white) easy to prove (grey) difficult to prove

Quantum theory of index formulas

Irving Fisher's "**five** tines fork"	
biased upwards	(2 +) uppermost, (1 +) mid-upper
neutral	(0) middle (unbiased)
biased downwards	(1 -) mid-lower, (2 -) lower most

Irving Fisher's **seven** grades quality scale (see **sec. 2.2**)
1. worthless, 2. weak, 3. correct, 4. good, 5. very good, 6. excellent, 7. superlative

A tentative list and grouping of axioms

Most of the controversies in index theory concerning the superiority or inferiority of certain index formulas are directly related to the **different significance authors attribute to the same axiom**. As will be shown below our (most positive) assessment of traditional formulas like Laspeyres and Paasche as opposed to Fisher' "ideal index" (vigorously advocated by other authors) is a consequence of much less emphasis we are willing to give to axioms like time reversibility or the so-called "quantity reversal test"[13] than others do.

[13] This is "test" T12 in **tab. 3.1.1**. Especially Diewert sets great store by these properties and he therefore rejects the traditional formulas. T12 for example rules out all formulas other than those in which quantities of 0 and t enter the formula in a symmetric manner (interestingly all so-called superlative indices are of precisely this type, i.e. complying with T12).

Note also that in addition to axioms that apply to the index itself axioms may be postulated apply to the **growth rate** of the index in question.

Table 3.1.1: List of "tests"/axioms of price index functions $P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$
(**B**.∗,**T**∗ refer to Diewert's list of 20 or 21 tests, and **F**∗ to Fisher's system of tests)

	Name of test	Comment
	Group B.1: Basic tests	
T1	Positivity	P_{0t} = P(...) and all constituent vectors are positive
T2	Continuity	P(...) is a continuous function of its vectors
F1	Determinate (determinateness) test or weak continuity axiom	if any scalar argument in P(...) tends to zero, then P tends to a unique positive real number
T3 **F2**	Identity $P_{00} = 1$ (or: Constant prices test)	$P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) = 1$ if for all i = 1, 2,..., n commodities $p_{it} = p_{i0}$ [1)]
T4	Fixed basket test (= Constant quantities test)	$P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_0) = V_{0t}$
	Group B.2: Homogeneity tests	
T5 **F3**	proportionality (strict version) in current prices $P(\mathbf{p}_0, \mathbf{q}_0, \lambda\mathbf{p}_0, \mathbf{q}_t) = \lambda$	if all prices move in proportion, so does the index, or: if all period t prices change λ-fold then the value of P is also changed by λ ($\lambda \in \mathbb{R}$)[2)]
F3a	weak version $P(\mathbf{p}_0, \mathbf{q}_0, \lambda\mathbf{p}_0, \mathbf{q}_0) = \lambda$	= F3 provided quantities do not change $\mathbf{q}_t = \mathbf{q}_0$
T6	Inverse proportionality in $\mathbf{p}_0$	$P(\lambda\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_0, \mathbf{q}_t) = 1/\lambda$
T7	Invariance to proportional changes in current quantities	$P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_0, \lambda\mathbf{q}_t) = P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_0, \mathbf{q}_t)$
T8	Invariance to proportional changes in base quantities	$P(\mathbf{p}_0, \lambda\mathbf{q}_0, \mathbf{p}_0, \mathbf{q}_t) = P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_0, \mathbf{q}_t)$

Note that Diewert's definition of "proportionality" resembles the notion of linear homogeneity:

(T5)	proportionality in current prices $P(\mathbf{p}_0,\mathbf{q}_0,\lambda\mathbf{p}_t,\mathbf{q}_t) = \lambda\, P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t)$	if all current period prices are multiplied by $\lambda > 0$ the new price index is λ times the old price ind. [3)]
(T6)	Inverse proportional. in prices $\mathbf{p}_0$	$P(\lambda\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) = 1/\lambda\, P(.\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$

	Group B.3: Invariance and symmetry tests	
T9	Commodity reversal test	invariance upon changes in the ordering of com.
T10 **F4**	Invariance to changes in the units of measurem. = commensurability	independence of the quantities to which price quotations refer (i.e. units of measurement)[4)]
T11 **F5**	Time/country reversal $P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)\, P(\mathbf{p}_t, \mathbf{q}_t, \mathbf{p}_0, \mathbf{q}_0) = 1$	interchanging $(\mathbf{q}_0,\mathbf{p}_0) \leftrightarrow (\mathbf{q}_t,\mathbf{p}_t)$, i.e. reversing the direction of comparison yields $P_{t0} = 1/P_{0t}$
T12	Quantity reversal test (quantities of both periods must enter symmetrically the index formula)	$P(\mathbf{p}_0, \mathbf{q}_t, \mathbf{p}_t, \mathbf{q}_0) = P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$ index remains invariant upon interchanging of quantity vectors
T13	Price reversal test PRT (obviously different from PRT in sec. 3.2)[5]	quantity index remains invariant upon interchanging of price vectors

Group B.4: Mean value tests		
T14	Mean value test for prices (often simply called: Mean value test)	P_{0t} lies between minimum and maximum price relative
T15	Mean value test for quantities	implicit Q_{0t} lies between min and max quantity relative
T16	Paasche + Laspeyres bounding test	$P^P \leq P_{0t} \leq P^L$ or $P^L \leq P_{0t} \leq P^P$

Group B.5: Monotonicity [6)] tests		
T17	Monotonicity in current prices	if any p_{it} increases ($p_{it} > p_{i0}$) P_{0t} increases ($P_{0t} > 1$)
T18	Monotonicity in base period prices	if any p_{i0} increases P_{0t} must decrease
T19	Monotonicity in current quantities	implicit Q_{0t} must increase if any q_{it} increases
T20	Monotonicity in base quantities	implicit Q_{0t} must decrease if any q_{i0} increases

Other tests, additivity (aggregative)[8)] properties		
T21 **F6**	Factor reversal $P_{0t}Q_{0t} = V_{0t}$	$Q_{0t} = P(\mathbf{q}_0,\mathbf{p}_0,\mathbf{q}_t,\mathbf{p}_t)$ if $P_{0t} = P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t)$ that is Q_{0t} is derived from P_{0t} by interchanging prices and quantities [7)]
F7	Circular test (see **sec. 3.2**)	also called transitivity test or chain test
F8	Withdrawal-and-entry test [9)] and equality test (see **sec. 5.2**)	Index should remain invariant if a price relative or sub-index is added or removed which is equal to the overall index
	Aggregative consistency of the index formula (see **sec. 5.2**)	Aggregation of relatives to subindices and subindices to the overall index follow same function [10)]
	Structural consistency of volumes (deflated values), SCV in **sec. 5.2**	Using P_{0t} as deflator should result in volumes that satisfy the same definitional equations values do

1) or: $P(\mathbf{p}_0,\mathbf{p}_t) = 1$ for an index **not** depending on quantities.
2) note that identity is obviously the special case $\lambda = 1$.
3) or: the price index function is (positively) homogenous of degree one in the components of the current period price vector $\mathbf{p}_t$.
4) We first referred to the commensurability test/axiom in connection with Dutot's index (see **sec. 1.2**).
5) According to Diewert the indices of Laspeyres (P^L) and Paasche fail this test while they are able to pass the differently defined PR-Test in **sec. 3.2**.
6) What is defined here is strictly speaking weak monotonicity as opposed to strict monotonicity
7) otherwise product test.
8) The purpose of these tests is to make sure that the overall-index can be compiled from sub-indices or be decomposed into sub-indices without difficulties and that aggregation and deflation yields reasonable results.
9) rarely mentioned at all; to be discussed in the appendix and (along with the equality test) in **sec. 5.2/4**.
10) with weights adjusted appropriately to the aggregation problem in question.

Commensurability can be expressed as follows

$$(3.1.2) \qquad P(\mathbf{L}\mathbf{p}_0, \mathbf{L}^{-1}\mathbf{q}_0, \mathbf{L}\mathbf{p}_t, \mathbf{L}^{-1}\mathbf{q}_t) = P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$$

where $\mathbf{L}$ is a $n \times n$ diagonal matrix with elements $\lambda_1,...,\lambda_n$, such that

$$\mathbf{L} = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{bmatrix} \text{and } \mathbf{L}^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & ... & 0 \\ 0 & 1/\lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1/\lambda_n \end{bmatrix}$$

When commensurability is satisfied the index function can be expressed in price relatives.

$$\mathbf{L} = \begin{bmatrix} 1/p_{10} & 0 & \dots & 0 \\ 0 & 1/p_{20} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1/p_{n0} \end{bmatrix}$$ with main diagonal elements $1/p_{i0}$.

Then we obtain

$\mathbf{Lp}_0 = \mathbf{1}$, where $\mathbf{1}' = [1\ 1\ \dots\ 1]$ and

$\mathbf{Lp}_t = \mathbf{a}$, the vector of price relatives $\mathbf{a}' = [p_{1t}/p_{10}\ \ p_{2t}/p_{20}\ \dots\ p_{nt}/p_{n0}]$. Furthermore

$\mathbf{L}^{-1}\mathbf{q}_0 = \mathbf{v}_0$ the vector of base period *values* $\mathbf{v}_0' =.[p_{10}q_{10}\ \ p_{20}q_{20}\ \dots\ p_{n0}q_{n0}]$ and

$\mathbf{L}^{-1}\mathbf{q}_t = \mathbf{v}_t$ the vector of *volumes* $\mathbf{v}_t' = [p_{10}q_{1t}\ \ p_{20}q_{2t}\ \dots\ p_{n0}q_{nt}]$.

(3.1.3) $P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t) = P(\mathbf{a},\mathbf{v}_0,\mathbf{v}_t)$,

By (***price***) ***dimensionality*** or homogeneity of degree 0 in prices

(3.1.4) $P(\lambda\mathbf{p}_0,\mathbf{q}_0,\lambda\mathbf{p}_t,\mathbf{q}_t) = P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t)$ (price) dimensionality.

In combination with commensurability quantity dimensionality is implied:

if price dimensionality and commensurability is met	→	then also quantity dimensionality

Quantity dimensionality (also called "weak commensurability ") is defined as follows

(3.1.5) $P(\lambda\mathbf{p}_0,\lambda^{-1}\mathbf{q}_0,\lambda\mathbf{p}_t,\lambda^{-1}\mathbf{q}_t) = P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_t,\mathbf{q}_t)$,

The idea of the "***identity*** test" has been introduced by Laspeyres. As already explained above

identity: if *no* price changes the price index function should be 1 (unity).

(3.1.6) strict identity $P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_0,\mathbf{q}_t) = 1$, if $\mathbf{p}_t = \mathbf{p}_0$

(3.1.6a) weak identity $P(\mathbf{p}_0,\mathbf{q}_0,\mathbf{p}_0,\mathbf{q}_0) = 1$, where $\mathbf{p}_t = \mathbf{p}_0$ *and* $\mathbf{q}_t = \mathbf{q}_0$

A statement to be regarded in a certain sense as the "opposite" of identity is:

if any *one* single price taken in isolation* is rising (or declining), the index function should *not* be 1 but indicate a rise or decline.

* hence also the case of *all* prices or *some* prices is covered.

This is guaranteed by the ***monotonicity*** (in current period prices) axiom. In order to prove that identity and monotonicity represent indeed two independent and different properties it should be demonstrated that at least one example of an index function exists that fits to field (1,2) and to field (2,1) respectively:

index	prices remain constant $p_{it} = p_{i0}$	prices going up/down
constant $P_{0t} = 1$	(1,1) identity	(1,2) $\Sigma p_0q_0/\Sigma p_0q_0 = 1$ = const.
indicates a change*	(2,1) $V_{0t} = \Sigma p_tq_t/\Sigma p_0q_0$	(2,2) monotonicity

* in the correct direction

(3.1.7) strict proportionality $P(\mathbf{p}_0,\mathbf{q}_0,\lambda\mathbf{p}_0,\mathbf{q}_t) = \lambda$, where $\lambda \in \mathbb{R}$, and $\mathbf{p}_t = \lambda\mathbf{p}_0$

(3.1.7a) weak proportionality $P(\mathbf{p}_0,\mathbf{q}_0,\lambda\mathbf{p}_0,\mathbf{q}_0) = \lambda$, where $\lambda \in \mathbb{R}$, $\mathbf{p}_t = \lambda\mathbf{p}_0$, $\mathbf{q}_t = \mathbf{q}_0$

Proportionality implies identity but not conversely:

if an index satisfies proportionality → then also identity

P^L, P^P, P^F, P^{DR}, P^{HPL}, P^W, P^{ME} and P^{GK} **all** satisfy the following tests:
1. identity, 2. determinateness, 3. commensurability, and 4. proportionality

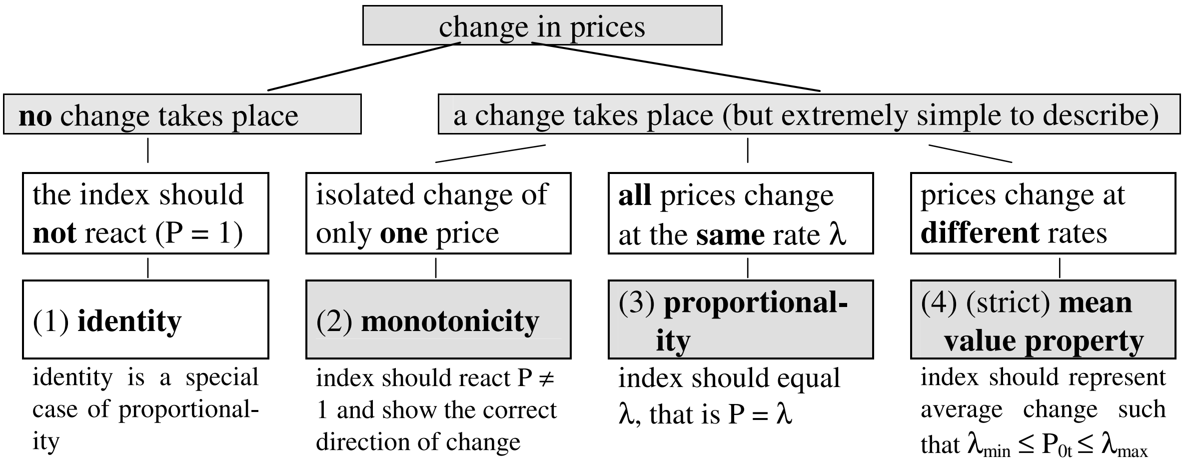
The following ***uniqueness theorem*** (**UT - 1)** is easy to verify:

A pair of Fisher indices P^F_{0t}, Q^F_{0t} is the only pair of indices that satisfies the product test (or factor reversal test) $P^*_{0t}Q^*_{0t} = V_{0t}$ *and* $P^*_{0t}/P^L_{0t} = Q^*_{0t}/Q^L_{0t}$

An example of an ***inconsistency theorem***

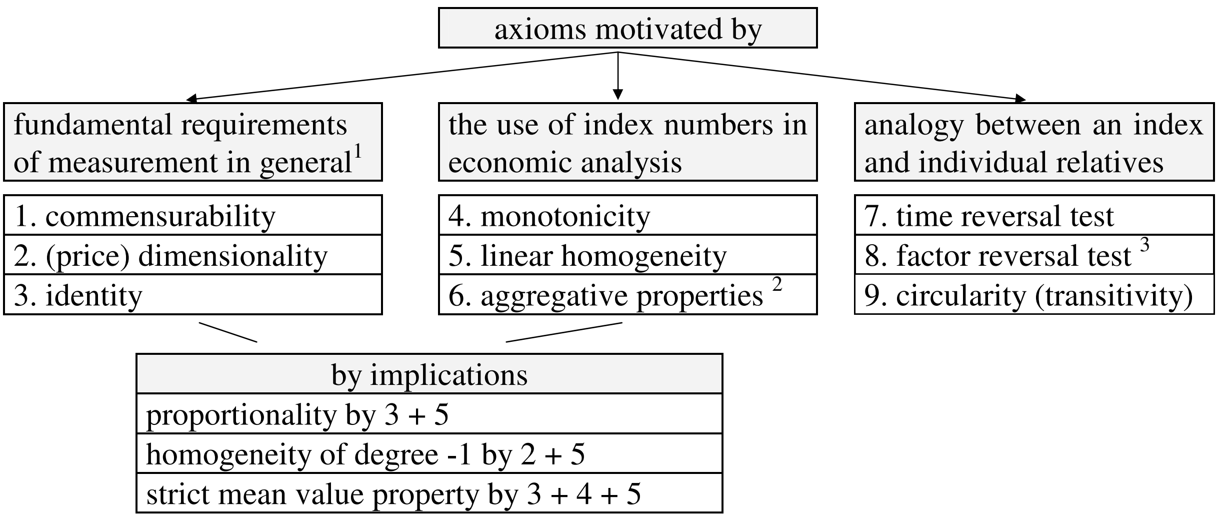
There do not exist functions, P_{0t} and Q_{0t} which satisfy simultaneously
1. the identity (strict or weak) axiom, 2. the circular test, and 3. the product test.

Figure 3.1.3: Relations among some axiomatic properties



Relations among the properties are for example: (3) → (1), and (4) → (3).

Figure 3.1.4: Tentative classification of axioms and their uses



1 they may be called invariance axioms
2 see **sec. 5.2**
3 unlike the factor reversal test the product test is deemed necessary

3.2. Fundamental axioms and their interpretation

a) Meaning of strict and weak monotonicity	f) The meaning of linear homogeneity
b) Additivity and multiplicativity	g) Linear homogeneity and proportionality
c) Generalization of Bortkiewicz's theorem*	h) Discrete time approximations and weights
d) Mean value property (for price relatives)	i) Proportionality of quantity indices**
e) Monotonicity, proport. mean value prop.	j) Value dependence test

*) for additive indices, ** "value index preserving test"

a) The meaning of strict and weak monotonicity

(3.2.1) $P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t^*, \mathbf{q}_t) > P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$, if $\mathbf{p}_t^* \geq \mathbf{p}_t$ and

(3.2.2) $P(\mathbf{p}_0^*, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) < P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$, if $\mathbf{p}_0^* \geq \mathbf{p}_0$.

In contrast to strict monotonicity the so called *weak monotonicity* is defined by

(3.2.1a) $P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) > P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_0, \mathbf{q}_t)$ if $\mathbf{p}_t \geq \mathbf{p}_0$ and

(3.2.2a) $P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) < P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_0, \mathbf{q}_t)$ if $\mathbf{p}_t \leq \mathbf{p}_0$.

Strict monotonicity implies weak monotonicity

if strict monotonicity is met → then also weak monotonicity

but the converse is not true. Two (independent) variants of weak monotonicity:

prices decline	prices rise ($p_t > p_0$), eq. 1a	
($p_t < p_0$), eq. 2a	yes	no
yes	Palgrave's index	P_{min}
no	P_{max}	median of price relatives

P_{min} and P_{max} are given by $\min\left(\frac{p_{it}}{p_{i0}}\right)$ and $\max\left(\frac{p_{it}}{p_{i0}}\right)$ respectively.

Figure 3.2.1: Strict and weak monotonicity

Monotonicity is a statement concerning

weak monotonicity	**strict** monotonicity
one index only, such that $P_{0t} > 1$ or $P_{0t} < 1$ is required	the comparison of **two** indices, P_{0t}^* and P_{0t} such that $P_{0t}^* > P_{0t}$ or $P_{0t}^* < P_{0t}$ is required
the index function is monotonically increasing in the price relatives*	P_{0t} and P_{0t}^* refer to situations which are partly the same and partly different

inequality	the same is	different is
eq. 3.2.1	$\mathbf{p}_0, \mathbf{q}_0, \mathbf{q}_t$	$\mathbf{p}_t$ and $\mathbf{p}_t^*$
eq. 3.2.2	$\mathbf{p}_t, \mathbf{q}_0, \mathbf{q}_t$	$\mathbf{p}_0$ and $\mathbf{p}_0^*$

* This simply means that we should get P > 1 (or P < 1) when the price relatives p_t/p_0 show a rise (or decrease) irrespective of whether the rise (decline) is due to rising prices p_t or lowered prices p_0 (or lowered prices p_t

and increased prices p_0). That is the reason why there is only one condition comparing p_t with p_0 whereas strict monotonicity needs two conditions (comparing p* with p in period t or in 0).

Index functions that can be conceived as means (averages) of price relatives are always monotonically increasing (decreasing) when the price relatives rise (decrease)*

* in other words: they are *by implication* monotonous in the *weak* sense

$$P_{0t}^{HB} = \left[\sum \frac{p_0}{p_t} \frac{p_0 q_0}{\sum p_0 q_0} \right]^{-1}$$ (harmonic mean with base year budget = harmonic Laspeyres)

$$P_{0t}^{PA} = \sum \frac{p_t}{p_0} \frac{p_t q_t}{\sum p_t q_t}$$ Palgrave's index.

Table 3.2.2: Two conditions of strict monotonicity

eq. 2 base	eq. 1 current period prices	
period prices	yes	no
yes	P^L, P^P etc.	Palgrave's index P^{PA}
no	harmonic Laspeyres P^{HB}	median of price relatives

b) Additivity and multiplicativity as special cases of strict monotonicity

Assume nonnegative price vectors, $\mathbf{p}_t^*$ and $\mathbf{p}_0^*$ which are defined as sums of two price vectors then the function P(...) is additive if

(3.2.3) $$P(\mathbf{p}_0, \mathbf{p}_t^*) = P(\mathbf{p}_0, \mathbf{p}_t) + P(\mathbf{p}_0, \mathbf{p}_t^+) = A + B$$ where $\mathbf{p}_t^* = \mathbf{p}_t + \mathbf{p}_t^+$, and

(3.2.4) $$\frac{1}{P(\mathbf{p}_0^*, \mathbf{p}_t)} = \frac{1}{P(\mathbf{p}_0, \mathbf{p}_t)} + \frac{1}{P(\mathbf{p}_0^+, \mathbf{p}_t)} = \frac{1}{C} + \frac{1}{D}$$ where $\mathbf{p}_0^* = \mathbf{p}_0 + \mathbf{p}_0^+$.

eq. 4	eq. 3 satisfied	eq. 3 violated
satisfied	P^L, P^P, Dutot P^D	unweighted harmonic mean of P^L and P^P
violated	Carli's index P^C index of Drobisch ½($P^L + P^P$)	Fisher's ideal index P^F

Weak variant of additivity: vector $\mathbf{p}_t^+ = \begin{bmatrix} b \\ \dots \\ b \end{bmatrix}$ and $\mathbf{p}_0^+$ is defined correspondingly.

(3.2.5) $$P(\mathbf{p}_0^*, \mathbf{p}_t^*) = P(\mathbf{K}\mathbf{p}_0, \mathbf{L}\mathbf{p}_t) = P(\mathbf{p}_0, \mathbf{p}_t) \cdot \phi(\kappa_1, \dots, \kappa_n, \lambda_1, \dots, \lambda_n)$$

where **K** and **L** are diagonal matrices $\mathbf{K} = \begin{bmatrix} \kappa_1 & & 0 \\ & \dots & \\ 0 & & \kappa_n \end{bmatrix}$ and $\mathbf{L} = \begin{bmatrix} \lambda_1 & & 0 \\ & \dots & \\ 0 & & \lambda_n \end{bmatrix}$

and ϕ is a function depending on the real numbers $\kappa_1, \kappa_2, \dots, \kappa_n, \lambda_1, \lambda_2, \dots, \lambda_n$ only such that ϕ is a positive real number. The logarithmic Laspeyres index is multiplicative in current period prices only (eq. 3.2.5). **Theorem**: An index function P( ) that satisfies the conditions of additivity necessarily must have the following form $P^A = \mathbf{a}'\mathbf{p}_t / \mathbf{b}'\mathbf{p}_0$. This explains also why for example Fisher's ideal index P^F does not fulfil the conditions of additivity. The same is true for the quadratic mean index[14] (3.2.8) $P_{0t}^{QM} = \sqrt{\sum \left(\frac{p_t}{p_0}\right)^2 \frac{p_0 q_0}{\sum p_0 p_0}}$.

c) Generalization of Bortkiewicz's theorem for additive indices

Figure 3.2.2: Generalisation of Bortkiewicz's theorem (ratio of two additive indices)

first additive index X_0	second additive index X_t
$X_0 = \frac{\sum x_t y_0}{\sum x_0 y_0}$	$X_t = \frac{\sum x_t y_t}{\sum x_0 y_t}$

↓ weighted (weights $w_0 = x_0 y_0 / \sum x_0 y_0$ throughout) arithmetic mean of the relatives ↓

x_t/x_0 : $\overline{X} = X_0$	y_t/y_0 : $\overline{Y} = \frac{\sum y_t x_0}{\sum y_0 x_0}$ *)

↓ variances (weights w_0) of the relatives ↓

x_t/x_0 : $s_x^2 = \sum \left(\frac{x_t}{x_0} - \overline{X}\right)^2 w_0$	y_t/y_0 : $s_y^2 = \sum \left(\frac{y_t}{y_0} - \overline{Y}\right)^2 w_0$

the covariance is given by
$s_{xy} = \sum \left(\frac{x_t}{x_0} - \overline{X}\right)\left(\frac{y_t}{y_0} - \overline{Y}\right) w_0 = \frac{\sum x_t y_t}{\sum x_0 y_0} - \overline{X} \cdot \overline{Y}$
and the ratio of two additive indices
(3.2.9) $\frac{X_t}{X_0} = 1 + r_{xy} V_x V_y = 1 + \frac{s_{xy}}{\overline{X} \cdot \overline{Y}}$ where $r_{xy} = \frac{s_{xy}}{s_x s_y}, V_x = \frac{s_x}{\overline{X}}$ and $V_y = \frac{s_y}{\overline{Y}}$

*) The formula of $\overline{Y}$ can be derived from $\overline{X} = X_0$ by interchanging x and y.

It can easily be seen that the special case of **sec. 1.3** was as follows: $X_0 = \overline{X} = P_{0t}^L, X_t = P_{0t}^P$ and $\overline{Y} = Q_{0t}^L$. Only in this case the coefficients of variation, V_x and V_y are symmetrically defined, one representing the relative dispersion of *price* relatives and the other the relative dispersion of *quantity* relatives.

d) Mean value property (mean value test for price relatives)

The index should take a value between the smallest and the largest price relative (a_{0t}^i)

(3.2.10) $\min\left(a_{0t}^i\right) \leq P_{0t} \leq \max\left(a_{0t}^i\right)$ (*strict* mean value property).

(3.2.11) $P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) = \lambda \min\left(a_{0t}^i\right) + (1-\lambda)\max\left(a_{0t}^i\right)$

where "strict" means $0 < \lambda < 1$ whilst in case of "weak" $0 \leq \lambda \leq 1$ is admitted.

[14] The formula P^{QM} will be referred to in **sec. 5.2** because it is aggregative consistent but not (more restrictive) additive in the sense defined above.

e) Relations between monotonicity, proportionality and mean value property

if monotonicity* and proportionality** is met	→	then also strict mean value property

* weak monotonicity sufficient
** or which is the same: identity and linear homogeneity

Again the converse relation is not true (if *strict* monotonicity is concerned at least). An example for this is once more Palgrave's index.

given strict mean value property	→	then also proportionality
given strict mean value property	→	then also weak monotonicity
given weak mean value property	→	then also proportionality

f) The meaning of linear homogeneity

(3.2.12) $$P(\mathbf{p}_0, \mathbf{q}_0, \lambda\mathbf{p}_t, \mathbf{q}_t) = \lambda P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t), \qquad \lambda \in \mathbb{R}$$

(3.2.13) $$P(\lambda\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t) = \frac{1}{\lambda} P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t),$$

given linear homogeneity and dimensionality	→	then also homogeneity of degree -1

(3.2.14) $$P_{0t}^{Y} = \sqrt{\frac{\sum p_t^2 q_t^2}{\sum p_0^2 q_0^2}},$$ (3.2.14a) $$P_{0t}^{Y*} = \sqrt{\frac{\sum p_t^2 q_0^2}{\sum p_0^2 q_0^2}}$$

g) Linear homogeneity and proportionality

if linear homogeneity and identity are met	→	then also (strict) proportionality

Linear homogeneity (LH) and (strict) proportionality (PR) are independent:

Table 3.2.3: Independence of linear homogeneity and proportionality

LH	PR	Examples
yes	no	P_{0t}^{Y} (as opposed to P_{0t}^{Y*}), value index V_{0t}
no	yes	P_{0t}^{EX} [1]; Stuvel's indices [2] (P_{0t}^{ST}, Q_{0t}^{ST}); Vartia-I index[3]; P_{0t}^{BA2} of Banerjee

1 exponential mean index (eq. 3.2.15 below) weighted or unweighted
2 only weak proportionality and identity but not linear homogeneity
3 This holds true for P^{V1} in contrast to the Vartia II index (P^{V2}). Olt 1996, p. 86 erroneously states that the Vartia II index violates linear homogeneity and the Vartia I index violates strict proportionality, see **sec. 2.6**.

(3.2.15) $$P_{0t}^{EX} = \ln\left[\frac{1}{n}\sum \exp\left(\frac{p_{it}}{p_{i0}}\right)\right],$$

h) Linear homogeneity, monotonicity and mean value property

if (strict) monotonicity, linear homogeneity and identity are met →	then also (strict) mean value property
if (strict) monotonicity and proportionality are met →	then also (strict) mean value property
if (strict) mean value property is met →	then also proportionality*

* Proportionality is clearly an implication of mean value property, but the converse is not true as can be seen by the index function P_{max} .

Figure 3.2.3: Linear homogeneity, monotonicity and proportionality

(see also **fig. 3.2.1**)

A statement concerning

one index only, reflecting correctly	difference of two indices
direction of change: **weak monotonicity**	direction: **strict monotonicity**
P_{0t} is increased ($P_{0t} > 1$) whenever any of the prices in t are raised or any of the prices in 0 are lowered	1. $P_{0t}^{*} > P_{0t}$ if $\mathbf{p}_t^{*} > \mathbf{p}_t$. or 2. $P_{0t}^{*} < P_{0t}$ if $\mathbf{p}_0^{*} > \mathbf{p}_0$ 2)
amount of change: **proportionality**	amount: **linear homogeneity**
if all prices rise with the same rate 1) λ, the index P_{0t} should amount to λ ($\mathbf{p}_t = \lambda \mathbf{p}_0$)	$P_{0t}^{*} = \lambda P_{0t}$ if prices in t are $\mathbf{p}_t^{*} = \lambda \mathbf{p}_t$ and in 0 the same, that is $\mathbf{p}_0^{*} = \mathbf{p}_0$ 3)

1) more precisely λ is the growth *factor* of prices.
2) and the other price vectors (p_0 in 1 and p_t in 2) remain unchanged
3) comparing prices $\mathbf{p}_t$ with $\mathbf{p}_0$ (proportionality) or prices $\mathbf{p}_t$* and $\mathbf{p}_t$ (lin. homogeneity)

i) Proportionality with respect to quantity indices, the "value index preserving test"

Proportionality in the case of a quantity index Q means $Q(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \lambda\mathbf{q}_0) = \lambda$, and when λ = 1 we should get Q = 1 (identity) and therefore

(3.2.16) $$P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \lambda\mathbf{q}_0)\, \underbrace{Q(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \lambda\mathbf{q}_0)}_{=\lambda=1} = V_{0t} = \frac{\sum p_t q_0}{\sum p_0 q_0} = P_{0t}^{L} = \frac{\sum p_t q_t}{\sum p_0 q_t} = P_{0t}^{P}$$

called "value index preserving test" by Vogts (not to be confounded with the Value dependence test).

Table 3.2.4: Summary information on relationships among axioms

assumption(s)	consequence
1 <ins>**C**</ins>ircular test + identity	time reversal test
3 Linear <ins>**H**</ins>omogeneity + identity	(strict) proportionality
5 strict <ins>**Me**</ins>an value property	weak monotonicity, proportionality, dimension. [1]
7 strict <ins>**Mo**</ins>noton. + proportionality[2]	strict mean value property

assumption(s)	consequence
2 <ins>**D**</ins>imensionality + commensurability	quantity dimensionality
4 Linear <ins>**H**</ins>omogeneity + dimensionality	homogeneity of degree -1
6 strict <ins>**Mo**</ins>notonicity	weak monotonicity additivity
8 <ins>**P**</ins>roportionality	identity (simply the special case λ = 1)

1 and of course also weak mean value property
2 because of 2 also strict monotonicity + linear homogeneity + identity → strict mean value property.

j) "Value dependence test", another uniqueness theorem for Fisher's ideal index

The function $P_{0t} = P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$, $\mathbb{R}^{4n}_{++} \to \mathbb{R}_{++}$ and the following function
$P_{0t} = f(\sum p_0 q_0, \sum p_0 q_t, \sum p_t q_0, \sum p_t q_t)$, $\mathbb{R}^{4}_{++} \to \mathbb{R}_{++}$
or simply f(a, b, c, d) should yield the same result.

This means that it should be possible to express the index function P_{0t} as a function of the four aggregates $\Sigma p_0 q_0$, $\Sigma p_0 q_t$, $\Sigma p_t q_0$, and $\Sigma p_t q_t$.

3.3. Systems of axioms

a) Irving Fisher's system of axioms (tests)	c) Two systems of Eichhorn and Voeller
b) A system of Marco Martini	d) Additional systems of B. Olt

a) Irving Fisher's system of axioms (tests)

F1	determinate (determinateness) test
F2	identity $P_{00} = 1$ ($p_{it} = p_{i0}$ for all i)
F3	commensurability
F4	proportionality (strict version) $(\mathbf{p}_0, \mathbf{q}_0, \lambda\mathbf{p}_0, \mathbf{q}_t) = \lambda$
F5	time/country reversal test $P_{t0} = 1/P_{0t}$
F6	factor reversal test $P_{0t}Q_{0t} = V_{0t}$
F7	circular test $P_{0t} = P_{0s}P_{st}$, or $P_{0t} = P_{0r}P_{rs}P_{st}$ etc. for ***all*** 0, r, s, t
F8	withdrawal-and-entry test

(3.3.1) $$P^F_{01}P^F_{12} = \sqrt{P^L_{01}P^P_{01}P^L_{12}P^P_{12}} = \sqrt{(P^L_{01}P^L_{12})(P^P_{01}P^P_{12})} \neq P^F_{02} = \sqrt{P^L_{02}P^P_{02}}$$

Besides *inconsistency* doubts also arose as to the *independence* of the requirements.

if the circular test and identity is satisfied → then also the time reversal test

if proportionality is satisfied (**F4**) → then also identity (**F2**)

b) A system of minimum requirements of an index by Marco Martini

Examples given by Martini to demonstrate the independence of this system

axioms fulfilled	axiom violated	example
2 and 3	1: identity	value index $V_{0t} = \Sigma p_t q_t / p_0 q_0$
1 and 3	2: commensurability	Dutot's index
1 and 2	3: linear homogeneity	exponential mean index (see **sec. 3.2**)

(3.3.2) $$P^{NM}_{0t} = \left(\frac{p_{1t}}{p_{10}}\right)^{\alpha_1}\left(\frac{p_{2t}}{p_{20}}\right)^{\alpha_2}\cdots\left(\frac{p_{nt}}{p_{n0}}\right)^{\alpha_n} \text{ where } \sum\alpha_i = 1, \alpha_1 < 0,\ \alpha_2,\ldots,\alpha_n > 0.$$

c) Two systems of axioms established by Eichhorn and Voeller

(3.3.4) $P_{0t} = P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$, $\mathbb{R}^{4n}_{++} \to \mathbb{R}_{++}$ (index function P).

Interesting features of the systems of Eichhorn and Voeller, both EV-4 and EV-5,

1. none of the ***reversal tests*** (time and factor reversal test) nor the circular test of Fisher is mentioned in the EV-systems
2. as in most other modern axiomatic systems also no mention is given to axioms restricting the type of ***weighting schemes*** wanted for an index, i.e. axioms dealing with quantities

   (Examples of such "axioms": weights of both periods, 0 and t should be used, and they should enter the formula in a symmetric fashion, more recent variable weights are to be preferred to constant weights of base period 0)
3. though monotonicity is an element of both EV-4 and EV-5 ***no*** attention has been given to ***additivity*** as a special case of monotonicity nor to other useful properties relating to ***aggregation*** and deflation,

Identity and linear homogeneity in EV-5 have been replaced in EV-4 by proportionality, being weaker and implied in EV-5. Hence

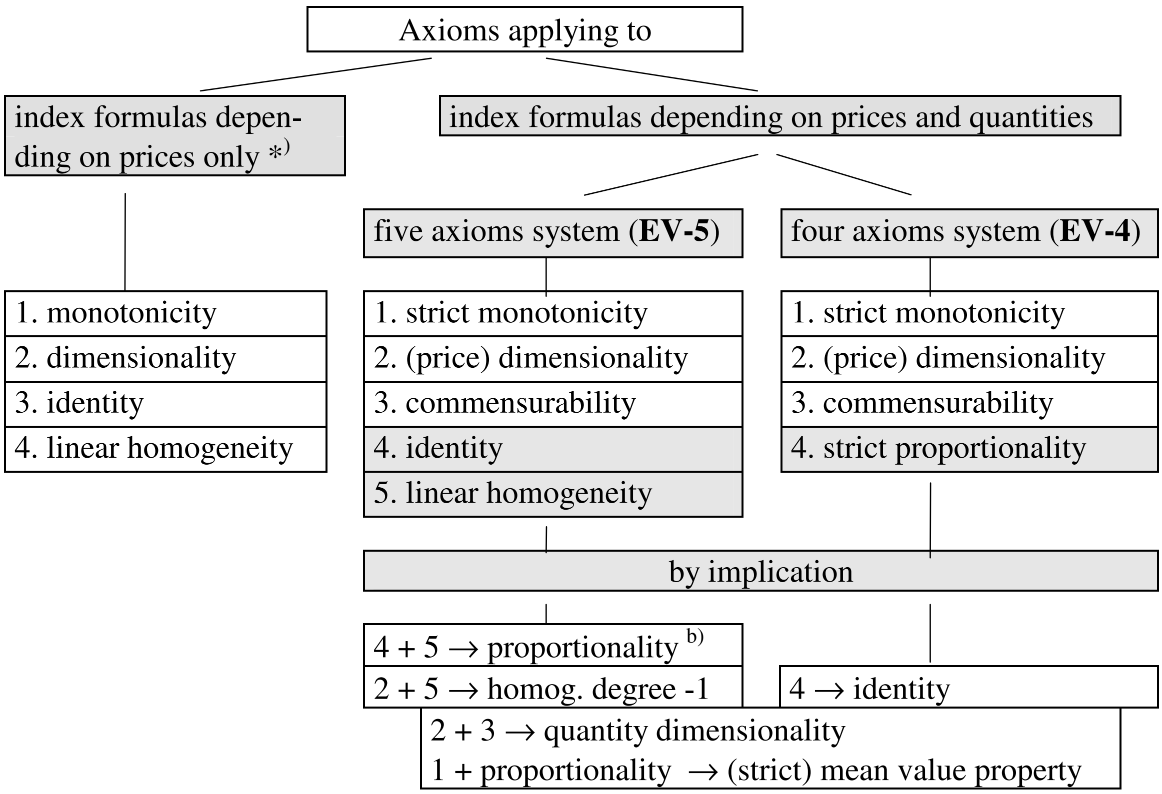
an index function satisfying the five axioms system (EV-5)	→	is also able to satisfy the four axioms system (EV-4)

Combinations of index formulas

If $P_1, P_2, ..., P_k$ are price indices each of them satisfying EV-4, or EV-5 respectively then $\tilde{P} = a_1P_1 + a_2P_2 + ... + a_nP_k$ or (more general) $\overline{P}$ will do so as well

(3.3.5) $$(\alpha_1 P_1^{\delta} + \ ... \ + \alpha_k P_k^{\delta})^{\frac{1}{\delta}} = \overline{P} \quad \begin{cases} \delta \neq 0, \quad \alpha_1 \geq 0, ..., \alpha_k \geq 0 \\ \alpha : \text{real constants}, \sum \alpha_i = 1 \end{cases}$$

Figure 3.3.1: Systems of axioms by Eichhorn & Voeller



d) Additional axiomatic systems of B. Olt and remarks on the choice between systems of axioms

Figure 3.3.2: Three systems of axioms by B. Olt

Three additional axiomatic systems

Olt 1	Olt 2	Olt 3
1. dimensionality	1. dimensionality	1. dimensionality
2. commensurability	2. commensurability	2. commensurability
3. weak monotonicity	3. weak monotonicity	3. strict mean value property
4. proportionality	4. weak mean value property	4. symmetry

An index function admissible in the definition "Olt 3" in contrast to "EV-4" is for example Palgrave's index because of being monotonous only in the weak sense (as [strict] mean value property implies weak monotonicity and proportionality) but not in the strict sense (that is the reason why this index does not comply with EV-4).

3.4. Log-change index numbers I: Cobb-Douglas- and Törnqvist-index

a) Growth rates, log changes, new formulas	c) The Törnqvist index
b) Cobb Douglas index, circular test	d) Quantitative relations between 6 indices

a) Growth rates, log changes, and new index formulas on the basis of log changes

(3.4.1) $r_t = \frac{y_t - y_{t-1}}{y_{t-1}} = \frac{\Delta y_t}{y_{t-1}}$, and

(3.4.2) $f_t = \frac{y_t}{y_{t-1}} = 1 + r_t$.

Growth rates and growth factors have the following *two disadvantages*:

- they are *not symmetric*, that is: $\frac{y_t - y_{t-1}}{y_{t-1}} \neq -\frac{y_{t-1} - y_t}{y_t}$ and
- the *sum* of two (or more) growth rates *over time, has no meaningful interpretation*.

Furthermore, a general notion of growth rate could be as follows:

(3.4.3) $\text{growth rate} = \frac{\text{absolute change}}{\text{level}} = \frac{\Delta y}{A(y)}$,

(1.4.8a) $D\ell_t = \ln\left(\frac{y_t}{y_{t-1}}\right) = \ln(y_t) - \ln(y_{t-1}) = \ln(f_t)$, and

(3.4.4) $L(y_t, y_{t-1}) = L(y_{t-1}, y_t) = \frac{y_t - y_{t-1}}{\ln(y_t / y_{t-1})}$ if $y_t \neq y_{t-1}$.

(3.4.5) $D\ell_t = \ln\left(\frac{y_t}{y_{t-1}}\right) = r_t^L = \frac{y_t - y_{t-1}}{L(y_t, y_{t-1})}$,

Growth factor f and growth rate r of P_{0t}^L, as an example are defined as follows

(3.4.6) $f(P^L_{0t}) = 1 + r(P^L_{0t}) = \frac{P^L_{0t}}{P^L_{0,t-1}} = \sum \frac{p_{i,t}}{p_{i,t-1}} \left(\frac{p_{i,t-1} q_{i0}}{\sum p_{i,t-1} q_{i0}} \right) = \sum \frac{p_{it}}{p_{i,t-1}} \beta_{it} = P_{t-1,t(0)}$ is an index with variable weights whereas both, P^L_{0t} and $P^L_{0,t-1}$ have the same constant weights $p_0q_0/\Sigma\, p_0q_0$.

Table 3.4.1: Advantages of log changes over traditional growth rates

aspect	log changes	traditional growth rates[1)]
symmetry	$\ln(y_t) - \ln(y_{t-1}) = - [\ln(y_{t-1}) - \ln(y_t)]$	no symmetry
summation over successive intervals	$D\ell_t + D\ell_{t+1} = \ln\left(\frac{y_{t+1}}{y_{t-1}}\right)$ is a growth related to a time span of *two* periods [2)]	the sum $r_t + r_{t+1}$ is not meaningful
eq. 3.4.3 interpretation	$D\ell_t = \frac{\Delta y}{A(y)} = \frac{y_t - y_{t-1}}{L(y_t, y_{t-1})}$ see eq. 3.4.5 log mean of y_t and y_{t-1} as "level"	lower (or upper) bound, that is y_{t-1} as level A(y) in the denominator

1) r_t in eq. 3.4.1
2) correspondingly the sum of m adjacent log-change-terms measures a change over m periods

Table 3.4.2: Definition of a "log-change" (price) index P^*_{0t}

The logarithm $\ln(P^*_{0t})$ of a log-change (price) index P^*_{0t} is a function of logarithmic price relatives $Da^i_{0t} = \ln(p_{it}/p_{i0})$; for example a weighted arithmetic mean of such Da^i_{0t} terms:

(3.4.8) $\ln(P^*_{0t}) = \sum g_i (Da^i_{0t})$ with weights g_i where $a^i_{0t} = p_{it}/p_{i0}$.:

To give some examples:

$\ln(P^*_{0t})$	P^*_{0t}	weights
$\frac{1}{n}\sum \ln\left(\frac{p_{it}}{p_{i0}}\right)$ (Carli-type)	$\prod\left(\frac{p_{it}}{p_{i0}}\right)^{1/n}$ (Jevons)	g_i = 1/n for all i
$\ln(DP^L_{0t}) = \sum_{i=1}^{n} \ln\left(\frac{p_{it}}{p_{i0}}\right) g_i$	$DP^L_{0t} = \prod_i \left(\frac{p_{it}}{p_{i0}}\right)^{g_i}$	$g_i = p_{i0}q_{i0} \big/ \sum p_{i0}q_{i0}$
DP^L_{0t} = logarithmic Laspeyres index*		

* The notation DP* is chosen in order to indicate a relationship between a traditional index P* and its "log change" counterpart.

See figure 3.4.1 (next page) for an overview over Log change indices

All index functions built as geometric means have extremely simple formulas of growth factors. For example the unweighted **Jevons** index $P^{JV}_{0t} = \prod_i \left(\frac{p_{it}}{p_{i0}}\right)^{1/n} = \sqrt[n]{\prod a^i_{0t}}$ meets transitivity. The growth factor of P^{JV}_{0t} is simply

(3.4.7) $f(P^{JV}_{0t}) = P^{JV}_{0t} / P^{JV}_{0,t-1} = \prod (p_{it}/p_{i,t-1})^{1/n}$,

the geometric mean of $p_{it}/p_{i,t-1}$ terms so that

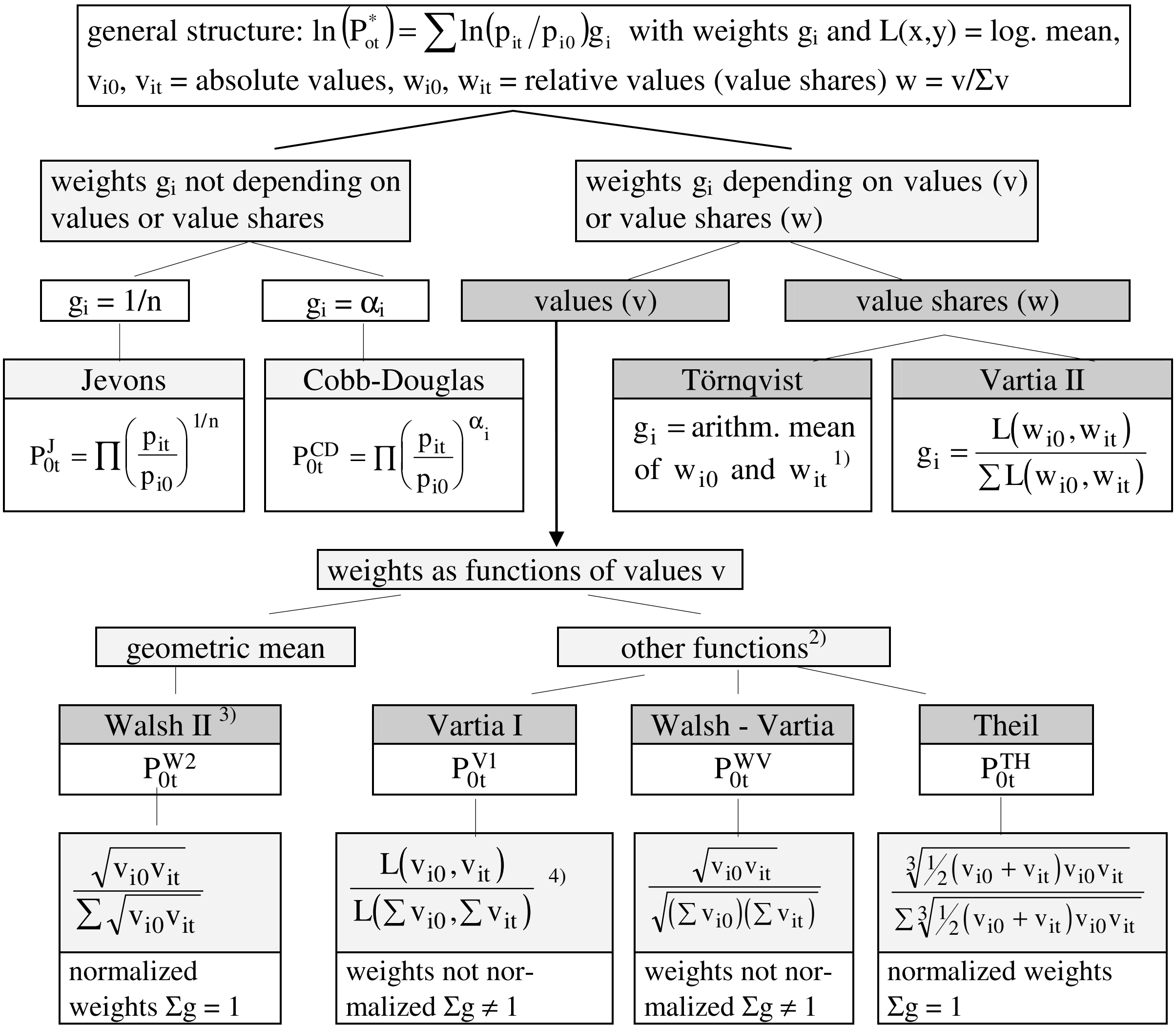
$\prod (p_{it}/p_{i0})^{1/n} = \prod (p_{i1}/p_{i0})^{1/n} \cdot \prod (p_{i2}/p_{i1})^{1/n} \ldots \prod (p_{it}/p_{i,t-1})^{1/n}$ or (equivalently)

(3.4.7a) $$Da^i_{0t} = \ln\left(\frac{p_{it}}{p_{i0}}\right) = \ln\left(\frac{p_{i1}}{p_{i0}}\right) + \ldots + \ln\left(\frac{p_{it}}{p_{i,t-1}}\right) = \sum_{\tau=1}^{\tau=t} \ln\left(\frac{p_{i,\tau}}{p_{i,\tau-1}}\right) = \sum_{\tau} \ln(\ell_\tau).$$

However for the Carli index we get

(3.4.7b) $$f(P^C_{0t}) = P^{JC}_{0t} / P^C_{0,t-1} = \frac{\sum (p_{it}/p_{i0})/n}{\sum (p_{i,t-1}/p_{i0})/n} \neq \sum (p_{it}/p_{i,t-1})/n .$$

Figure 3.4.1: Log-change indices



1) $\overline{w}_i = (w_{it} + w_{it})/2$

2) an index function in this context not mentioned here is the index of Rao

3) The name was given because this index has some resemblance to the "normal" index of Walsh $P^W_{0t} = P^{W1}_{0t} = \frac{\sum p_{it}\sqrt{q_{i0}q_{it}}}{\sum p_{i0}\sqrt{q_{i0}q_{it}}} = \sum \frac{p_{it}}{p_{i0}} \frac{\sqrt{(p_{i0}q_{i0})(p_{i0}q_{it})}}{\sum\sqrt{(p_{i0}q_{i0})(p_{i0}q_{it})}}$.

4) Note that in general $\sum_i L(v_{i0}, v_{it}) \neq L(\sum v_{i0}, \sum v_{it},)$.

b) Cobb Douglas index P^{CD}_{0t}, constant weights and the circular test

(3.4.8) $$P^{CD}_{0t} = \prod_{i=1}^{n} \left(\frac{p_{it}}{p_{i0}}\right)^{\alpha_i} \text{ (and } Q^{CD}_{0t} = \prod_{i=1}^{n} \left(\frac{q_{it}}{q_{i0}}\right)^{\alpha_i} \text{ correspondingly),}$$

where α_i are any real constants, $\Sigma\alpha_i = 1$ and $0 \le \alpha_i \le 1$, not necessarily expenditure shares.

(3.4.9) $$P_{02}^{CD} = \left[\left(\frac{p_{11}}{p_{10}}\right)^a \left(\frac{p_{21}}{p_{20}}\right)^{1-a}\right]\left[\left(\frac{p_{12}}{p_{11}}\right)^a \left(\frac{p_{22}}{p_{21}}\right)^{1-a}\right] = P_{01}^{CD} P_{12}^{CD}, \ (\alpha_1 = a,\ \alpha_2 = 1\text{-}a)$$

Assume constant growth factors of the prices $\lambda_1 = \frac{p_{1t}}{p_{1,t-1}}$ and $\lambda_2 = \frac{p_{2t}}{p_{2,t-1}}$ then the growth factor of P_{0t}^{CD} is $\lambda_1^a \lambda_2^{1-a}$, and *is constant for all periods t*. By contrast he same conditions prevailing in the case of the Laspeyres price index P_{0t}^L will result in the following growth factors:

$$P_{00}^L = 1 \to P_{01}^L : \sum \lambda_i \frac{p_{i0}q_{i0}}{\sum p_{i0}q_{i0}} = \sum \lambda_i \beta_{i1}, \quad P_{01}^L \to P_{02}^L : \quad \sum \lambda_i \frac{\lambda_i p_{i0}q_{i0}}{\sum \lambda_i p_{i0}q_{i0}} = \sum \lambda_i \beta_{i2}$$

$$P_{02}^L \to P_{03}^L : \quad \sum \lambda_i \frac{\lambda_i^2 p_{i0}q_{i0}}{\sum \lambda_i^2 p_{i0}q_{i0}} = \sum \lambda_i \beta_{i3}$$ and so on,

The growth factor of P^{CD} is a geometric mean with constant weights α_i (for all periods t = 1,2, ...) and therefore constant as well whereas the growth factor of P^L is an arithmetic mean with changing weights and tending to the largest individual growth factor of prices.

Example 3.4.1

Consider two commodities, with base period expenditure shares $w = w_1 = p_{10}q_{10} / \sum p_{i0}q_{i0}$ = 0.6 and $w_2 = 1 - w = 0.4$. Prices are increasing at a constant rate of 80% or 20% respectively such that the constant growth factors are $\lambda_1 = 1.8$ and $\lambda_2 = 1.2$, and the prices are $p_{1t} = \lambda_1^t p_{10}$ and $p_{2t} = \lambda_2^t p_{20}$ respectively. The series of P_{0t}^L now is determined by

	t = 0	t = 1	t = 2	t = 3	t = 4
P_{0t}^L	1	1.56	2.52	4.1904	7.128
$P_{0t}^L / P_{0,t-1}^L$		1.56	1.62	1.663	1.701

By contrast the growth rate of P_{0t}^{CD} is constant $(\lambda_1)^w (\lambda_2)^{1-w}$ = 1.5305.

The circular test and a characterization (uniqueness theorem) of P^{CD}

(3.4.10) $$P_{0s}^{LW} P_{st}^{LW} = \frac{\sum p_s q}{\sum p_0 q} \frac{\sum p_t q}{\sum p_s q} = P_{0t}^{LW} = \frac{\sum p_t q}{\sum p_0 q}$$

UT-7: The Cobb Douglas index is the *unique* index function that satisfies
1. the ***circular test*** (transitivity) ***and***
2. the following five fundamental axioms (EV-5): 1. monotonicity, 2. price-dimensionality, 3. linear homogeneity, 4. identity and 5. commensurability.

c) The Törnqvist index P_{0t}^T an "unbiased" index formula in a system of six indices

(3.4.11) $$P_{0t}^T = \prod_{i=1}^{n} \left(\frac{p_{it}}{p_{i0}}\right)^{\overline{w}_i}$$ where $\overline{w}_i$ is the mean of expenditure shares for period 0 and period t $\overline{w}_i = \frac{1}{2}(w_{i0} + w_{it}) = \frac{1}{2}\left(\frac{p_0 q_0}{\Sigma p_0 q_0} + \frac{p_t q_t}{\Sigma p_t q_t}\right)$, or alternatively

(3.4.12) $\ln(P_{0t}^T) = \frac{1}{2}\left[\Sigma w_{i0} \ln(p_{it} / p_{i0}) + \Sigma w_{it} \ln(p_{it} / p_{i0})\right] = \frac{1}{2}\left[\ln(DP_{0t}^L) + \ln(DP_{0t}^P)\right]$,

or of course equivalently $\log P_{0t}^T = \frac{1}{2}\left[\log(DP_{0t}^L) + \log(DP_{0t}^P)\right]$, where DP_{0t}^L denotes the "logarithmic Laspeyres price index", and DP_{0t}^P the "log. Paasche price index " respectively.

(3.4.13a)	either $P_{0t}^L > P_{0t}^F > P_{0t}^P$ or $P_{0t}^L < P_{0t}^F < P_{0t}^P$, and
(3.4.13b)	either $DP_{0t}^L > P_{0t}^T > DP_{0t}^P$ or $DP_{0t}^L < P_{0t}^T < DP_{0t}^P$.

As $P_{0t}^F = \sqrt{P_{0t}^L P_{0t}^P}$ is arising from (13a), so does (3.4.12a) $P_{0t}^T = \sqrt{DP_{0t}^L DP_{0t}^P}$ from the second (13b). Note that for example $P_{0t}^L > P_{0t}^P$ does not entail $DP_{0t}^L > DP_{0t}^P$ (**ex. 3.4.2**).

P^T is the geometric mean of two log-change indices DP^L and DP^P, much in the same way as P^F is the geometric mean of P^L and P^P. Thus the role P^T plays with respect to log-change indices (logarithms of price relatives) is similar to the role P^F plays in the case of normal indices (price relatives). There are however limitations to the analogy: P^T is a mean of relatives; P^F has no such mean-of-relatives interpretation[15]. On the other hand P^F passes the factor reversal test (is "ideal") while P^T is not even conforming to the (weaker) product test.

(3.4.14) $P_{0t}^T P_{t0}^T = \prod\left(\frac{p_{it}}{p_{i0}}\right)^{\overline{w}_i} \prod\left(\frac{p_{i0}}{p_{it}}\right)^{\overline{w}_i} = 1.$

(3.4.14a) $DP_{0t}^L DQ_{0t}^P = \prod_{i=1}^{n}\left(\frac{p_{it}}{p_{i0}}\right)^{w_{i0}} \prod_{i=1}^{n}\left(\frac{q_{it}}{q_{i0}}\right)^{w_{it}} \neq V_{0t}$,

Törnqvist's index does *not* meet the *factor reversal* test[16]

d) Quantitative relations between six indices

Numerical example (ex. 3.4.2)

commodity	base period (0)			current period (t)		
	price	quantity	value	price	quantity	value
A	10	20	200	12	20	240
B	20	16	320	18	20	360
C	16	30	480	24	25	600

The price relatives are for commodity A: 1.2, for B: 0.9 and for C: 1.5. Hence we get $DP^L = 1.2^{0.2}\ 0.9^{0.32}\ 1.5^{0.48} = 1.21820$, and $DP^P = 1.2^{0.2}\ 0.9^{0.3}\ 1.5^{0.5} = 1.23071$ and the two "traditional" indices $P^L = 1.248$, and $P^P = 1.2$. Interestingly:

though $P^P = 1.2 < P^L = 1.248$ we have DP^P 1.231 > $DP^L = 1.218$.

[15] Note also that the average weights used in P^T can be viewed as *crossing of weights* whereas eq. 3.4.12/12a can be regarded as *crossing of formulas*. Thus P^T has two interpretations (in terms of "crossing"), P^F only one.

[16] It does not even meet the product test since the implicit (indirect, antithetic, cofactor) quantity index V/P^T is not proportional in the quantities q_{it} (because they affect w_{it}).

The two geometric means are $P^T = \sqrt{DP^L DP^P} = 1.22444$ and $P^F = 1.22376$. For the harmonic Laspeyres index we get $P^{HL} = \left(\frac{0.2}{1.2} + \frac{0.32}{0.9} + \frac{0.48}{1.5}\right)^{-1} = 1.18734$ and for Palgrave's index $P^{PA} = 1.2 \cdot 0.2 + 0.9 \cdot 0.3 + 1.5 \cdot 0.5 = 1.26$, so that the third unbiased index turns out to be $\sqrt{P^{PA} P^{HL}} = 1.22313$.

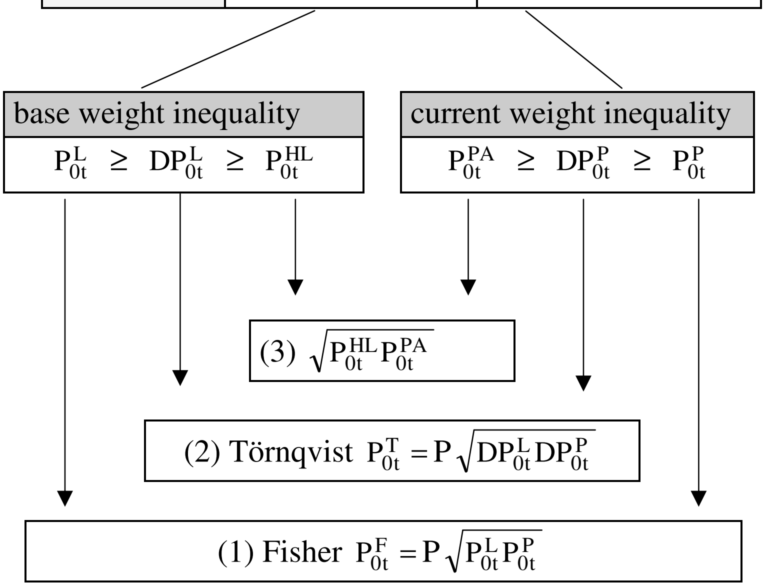
$P^P = 1.2$	$DP^P = 1.23$	$P^{PA} = 1.26$
$P^{HL} = 1.187$	$D^{PL} = 1.218$	$P^L = 1.248$

$Q^P = 0.962$	$DQ^P = 0.976$	$Q^{PA} = 0.992$
$Q^{HL} = 0.969$	$D^{QL} = 0.984$	$Q^L = 1.0$

Figure 3.4.2: A system of six index formulas (Vartia) and the "five tined fork" of I. Fisher

type of mean	type of weight structure w_{i0}	type of weight structure w_{it}
arithmetic	P^L (Laspeyres)	P^{PA} (Palgrave)
geometric	DP^L (log. Lasp.)	DP^P (log. Paasche)
harmonic	P^{HL} (harm.Lasp)	P^P (Paasche)

The five tined fork	
tine	formula
uppermost (+2)	P^{PA}
mid-upper (+1)	DP^P
middle (0)	P^L, P^T
unbiased	P^F, P^P
mid lower (-1)	DP^L
lower most (-2)	P^{HL}



Fisher's result was
$P^{PA} > DP^P > P^L > P^T >$ $P^F > P^P > DP^L > P^{HL}$

There was hardly any attention given to index no. 3. All three indices (1), (2) and (3) are "unbiased" index formulas and their results are in general in close agreement with one another.

Relations (Bortkiewicz type) between DP^P and DP^L (just like between P^P and P^L)

covariance between log changes in prices and volumes $\text{cov}(\dot{p}, \dot{v})$ or between log changes in prices and quantities $\text{cov}(\dot{p}, \dot{q})$).

(3.4.16a) $\log(DP^P_{0t}) - \log(DP^L_{0t})$ is *approximately* equal to $\text{cov}(\dot{p}, \dot{v})$,

(3.4.16b) $\log(P^P_{0t}) - \log(P^L_{0t}) \approx \text{cov}(\dot{p}, \dot{q})$.

3.5. Log-change index numbers II: Vartia's index formulas

a) Aggregation of log changes and the logarithmic mean

Figure 3.5.1: Aggregation growth rates over commodities (additive model)

Notation 1. consider change from 0 to t,
2. variable referring to individual commodity y_i ($i = 1, \ldots, n$),
3. aggregate $Y_\tau = \sum y_{i\tau}$, $\tau = 0, 1$ (additive model)

conventional growth rate	growth rate on the basis of log changes
(3.5.2) $r(Y) = \frac{Y_t - Y_0}{Y_0} = \sum_{i=1}^{n} a_{i0}\left(\frac{y_{it} - y_{i0}}{y_{i0}}\right) = \sum a_{i0} r(y_i)$	(3.5.3) $\ln\left(\frac{Y_t}{Y_0}\right) = \sum_{i=1}^{n} \frac{b_i}{\sum b_i} \ln\left(\frac{y_{it}}{y_{i0}}\right) = \sum \beta_i \ln\left(y_{it}/y_{i0}\right)$, $\beta_i = b_i/\Sigma b_i$
weights $a_{i0} = \frac{y_{i0}}{\sum y_{i0}} = \frac{y_{i0}}{Y_0}$	weights $b_i = L\left(a_{i0}, a_{it}\right)$, $a_{it} = \frac{y_{it}}{\sum y_{it}}$

Commentary on the type of weights

weights a_{i0} are making use of the structural information related to period 0 only	b_i being an average of a_{i0} and a_{it} are taking both periods into account*

* weights are "balanced", i.e. they employ structural data of both periods.

Decompositions of total value log changes

a) use means of <u>absolute</u> values v and weights $L(v_{i0}, v_{it})$, which leads to

$\sum_i \ln\left(\frac{v_{it}}{v_{i0}}\right) L(v_{i0}, v_{it}) = V_t - V_0$, and since $L\left(V_t, V_0\right) = \frac{V_t - V_0}{\ln\left(V_t/V_0\right)}$ we get

(3.5.4) $$\ln\left(\frac{V_t}{V_0}\right) = \sum \ln\left(\frac{v_{it}}{v_{i0}}\right) \frac{L(v_{it}, v_{i0})}{L(V_t, V_0)}$$ which is the basis for the <u>Vartia-I</u> index;

b) use value <u>shares</u> $w_{it} = v_{it}/\Sigma v_{it} = v_{it}/V_t$ and $w_{i0} = v_{i0}/\Sigma v_{i0} = v_{i0}/V_0$ and weights $L(w_{i0}, w_{it})$

$L\left(w_{it}, w_{i0}\right) = \frac{w_{it} - w_{i0}}{\ln\left(v_{it}/v_{i0}\right) - \ln\left(V_t/V_0\right)}$ to get

(3.5.5) $$L\left(w_{it}, w_{i0}\right)\ln\left(v_{it}/v_{i0}\right) = \left(w_{it} - w_{i0}\right) + L\left(w_{it}, w_{i0}\right)\ln\left(V_t/V_0\right)$$

and since $\sum w_{it} = \sum w_{i0} = 1$ we get

$\ln\left(\frac{V_t}{V_0}\right)\sum L(w_{it}, w_{i0}) = \sum\left(L(w_{it}, w_{i0})\ln\left(\frac{v_{it}}{v_{i0}}\right)\right)$, and finally

(3.5.6) $$\ln\left(\frac{V_t}{V_0}\right) = \sum\left(\ln\left(\frac{v_{it}}{v_{i0}}\right)\frac{L(w_{it}, w_{i0})}{\sum L(w_{it}, w_{i0})}\right)$$, which is the basis for the <u>Vartia-II</u> index.

b) The Vartia-I index $\left(P_{0t}^{V1}\right)$

The Vartia-I price index is defined as

(3.5.8) $$\ln\left(P_{0t}^{V1}\right) = \frac{\sum_{i=1}^{n} L\left(v_{it}, v_{i0}\right)\ln\left(\frac{p_{it}}{p_{i0}}\right)}{L\left(V_t, V_0\right)}$$, and correspondingly the quantity index

(3.5.8a) $$\ln\left(Q_{0t}^{V1}\right) = \sum_{i=1}^{n} L\left(v_{it}, v_{i0}\right)\ln\left(\frac{q_{it}}{q_{i0}}\right) \Big/ L\left(V_t, V_0\right),$$

(3.5.8b) $$\frac{L(v_{it}, v_{i0})}{L(V_t, V_0)}\ln\left(\frac{q_{it}}{q_{i0}}\right) = \frac{L(v_{it}, v_{i0})}{L(V_t, V_0)}\frac{q_{it} - q_{i0}}{L(q_{it}, q_{i0})} = \Delta\left(\ln\left(Q_{0t}^{V1}\right)\right)$$

c) The Vartia-II index $\left(P_{0t}^{V2}\right)$

(3.5.9) $\ln V_{0t} = \Sigma \hat{w}_i \ln\left(v_{it} / v_{i0}\right)$,

$\hat{w}_t = L\left(w_{it}, w_{i0}\right) / \sum L\left(w_{it}, w_{i0}\right)$ and $w_{it} = v_{it} / V_t = p_{it}q_{it} / \sum p_{it}q_{it}$ (w_{i0} defined analogously). By virtue of eq. 9 the indices P_{0t}^{V2} and Q_{0t}^{V2} pass the factor reversal test. To prove eq. 9 use (3.5.10) $\ln\frac{w_t}{w_0} = \ln\frac{v_t}{v_0} - \ln\frac{\Sigma v_t}{\Sigma v_0} = \ln\frac{v_t}{v_0} - \ln\frac{V_t}{V_0}$ taking into account $\Sigma w_t = \Sigma w_0 = 1$..

The Vartia II indices now are defined as follows

(3.5.11) $$\ln\left(P_{0t}^{V2}\right) = \Sigma\frac{L\left(w_{it}, w_{i0}\right)\ln\left(\frac{p_{it}}{p_{i0}}\right)}{\sum L\left(w_{it}, w_{i0}\right)} = \sum \hat{w}_{it} \ln\left(p_{it}/p_{i0}\right)$$, and

(3.5.11a) $$\ln\left(Q_{0t}^{V2}\right) = \sum \hat{w}_{it} \ln\left(q_{it}/q_{i0}\right).$$

Figure 3.5.2: Törnqvist, Vartia-II and Walsh-II index

general structure: $P(g) = \prod\left(\frac{p_{it}}{p_{i0}}\right)^g$

with weights g ($\Sigma g = 1$)

Törnqvist	Vartia II	Walsh II
$g = \overline{w} = \frac{1}{2}\left(w_0 + w_t\right)$	$g = \hat{w} = L(w_0, w_t) / \sum L(w_0, w_t)$	$g = \sqrt{v_0 v_t} / \Sigma\sqrt{v_0 v_t}$

$\sqrt{w_0 w_t} \leq L(w_0, w_t) \leq \frac{1}{2}\left(w_0 + w_t\right) = \overline{w}$, $\sqrt{w_0 w_t} = \sqrt{v_0 v_t} \Big/ \sqrt{V_0 V_t}$

d) Properties of Vartia indices

P^{V1} can (unlike P^{V2}) fail proportionality as well as linear homogeneity (see **ex. 3.5.2**).
Log change indices may violate monotonicity (**ex. 3.5.4**).

Example 3.5.4

	$\mathbf{p}_0$	$\mathbf{p}_t$	$\mathbf{q}_0$	$\mathbf{q}_t$
1	30	40	50	20
2	80		4	40

The following values are examined for the price p_{2t} 1, 5, 10, yielding values for various log change index functions as follows. We expect of course that p_{2t} = 5 will yield a higher P_{0t} as p_{2t} = 1 and again p_{2t} = 10 a higher value of P_{0t} as in the case of p_{2t} = 5. However this is not true for the following indices[17]: P_{0t}^{W2} = Walsh II, P_{0t}^{V1} = Vartia I, P_{0t}^{V2} = Vartia II, P_{0t}^{WV} = Walsh-Vartia, P_{0t}^{T} = Törnqvist, and P_{0t}^{TH} = Theil.

p_{2t}	P_{0t}^{L}	P_{0t}^{P}	P_{0t}^{W2}	P_{0t}^{V1}	P_{0t}^{V2}	P_{0t}^{WV}	P_{0t}^{T}	P_{0t}^{TH}
1	1.101	0.221	0.861	0.701	0.837	0.864	0.791	0.813
5	1.110	0.263	0.751	0.454	0.751	0.751	0.750	0.753
10	1.121	0.316	0.744	0.193	0.740	0.748	0.730	0.749

Only the two indices added for sake of comparison that is Laspeyres and Paasche (shaded area) comply with strict monotonicity, but *none* of the log change indices does so. Furthermore these indices differ tremendously from one another. So virtually *all* log-change indices display a decrease in prices when an isolated rise in price p_{2t} takes place from 1 to 5 and to 10. ♦

3.6. Ideal index functions and Theil's Best Linear Index (BLI)

Three-component model of value change (the structural component)

1. Additive model (value change)

(3.6.1) $$\mathbf{p}_t'\mathbf{q}_t - \mathbf{p}_t'\mathbf{q}_t = \mathbf{q}_0'(\mathbf{p}_t - \mathbf{p}_0) + \mathbf{p}_0'(\mathbf{q}_t - \mathbf{q}_0) + (\mathbf{q}_t' - \mathbf{q}_0')(\mathbf{p}_t - \mathbf{p}_0)$$
$$= \mathbf{q}_0'\Delta\mathbf{p}_t + \mathbf{p}_0'\Delta\mathbf{q}_t + \Delta\mathbf{q}_t'\Delta\mathbf{p}_t = \text{PC} + \text{QC} + \text{SC}.$$

The following interpretation of this simple definitional equation is usually given

- the pure price component (PC) is represented by $\mathbf{q}_0'\Delta\mathbf{p}_t = \sum p_{it}q_{i0} - \sum p_{i0}q_{i0}$
- the pure quantity component (QC) is $\mathbf{p}_0'\Delta\mathbf{q}_t = \sum p_{i0}q_{it} - \sum p_{i0}q_{i0}$
- the structural component (SC) is $\Delta\mathbf{q}_t'\Delta\mathbf{p}_t = \Delta\mathbf{p}_t'\Delta\mathbf{q}_t = \sum(p_{it} - p_{i0})(q_{it} - q_{i0})$.

Dividing both sides of eq. 1 by $\mathbf{p}_0'\mathbf{q}_0$ yields an equation known from Stuvel's approach:

(3.6.2) $$V - 1 = (P^L - 1) + (Q^L - 1) + (V - P^L - Q^L + 1) = \text{PC} + \text{QC} + \text{SC}.$$[18]

2. Multiplicative model (value ratio)

Aggregation of a two components multiplicative micro-model: Vartia's solution

Ideal index functions on the basis of log changes, like the two Vartia indices cannot be derived as easily as Fisher's indices, P_{0t}^{F} and Q_{0t}^{F}. The trick is to find appropriate weights for individual log changes.

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[17] Some log change indices not discussed above are also included in this example and Olt's original table.

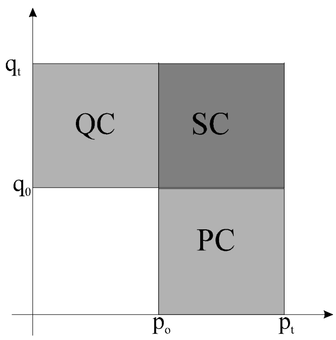
[18] **Fig. 3.6.1** provides a geometric interpretation of this relationship with three shaded fields representing PC, QC and SQ and shows how Stuvel has managed to allocate parts of the structural component SC to PC and QC.

Figure 3.6.1: Stuvel's way of deriving an ideal index

a) Additive approach

(3.6.2) $V - 1 = (P^L - 1) + (Q^L - 1) + (V - P^L - Q^L + 1)$

$= PC + QC + SC$



b) Stuvel's solution

price component PC	structural change component SC	quantity component QC
$P_{0t}^L - 1$	$V_{0t} - P_{0t}^L - Q_{0t}^L + 1$	$Q_{0t}^L - 1$

$\frac{1}{2}\left(V_{0t} - P_{0t}^L - Q_{0t}^L + 1\right)$ → new price component; $\frac{1}{2}\left(V_{0t} - P_{0t}^L - Q_{0t}^L + 1\right)$ → new quantity component

new price component $P^* = PC + ½SC$	new quantity component $Q^* = QC + ½SC$
$P^* = \frac{1}{2}\left(P_{0t}^L - Q_{0t}^L\right) + \frac{1}{2}\left(V_{0t} - 1\right)$	$Q^* = \frac{1}{2}\left(Q_{0t}^L - P_{0t}^L\right) + \frac{1}{2}\left(V_{0t} - 1\right)$

The product P*Q* is not meaningful, since $P^*Q^* = [(V - 1)^2 - (P^L - Q^L)^2]$. A better solution is

$P^{ST} = \frac{1}{2}\left(P_{0t}^L - Q_{0t}^L\right) + \frac{1}{2}R$ $\qquad$ $Q^{ST} = \frac{1}{2}\left(Q_{0t}^L - P_{0t}^L\right) + \frac{1}{2}R$

The product $P^{ST}Q^{ST}$ should equal V, thus setting $P^{ST}Q^{ST} = [R^2 - (P^L - Q^L)^2] = V$ we get

$$\frac{R}{2} = \sqrt{\left(\frac{P_{0t}^L - Q_{0t}^L}{2}\right)^2 + V_{0t}} = \sqrt{\left(\frac{Q_{0t}^L - P_{0t}^L}{2}\right)^2 + V_{0t}}$$

Figure 3.6.2: Fisher's way of deriving an ideal index

price component PC*	structural change component SC*	quantity component QC*
$\frac{\mathbf{p}_t'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_0} = P_{0t}^L$	$\frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_t'\mathbf{q}_0}\frac{\mathbf{p}_0'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_t} = 1 - \frac{C}{V_{0t}}$	$\frac{\mathbf{q}_t'\mathbf{p}_0}{\mathbf{q}_0'\mathbf{p}_0} = Q_{0t}^L$

$\sqrt{\frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_t'\mathbf{q}_0}\frac{\mathbf{p}_0'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_t}}$ → new price component; $\sqrt{\frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_t'\mathbf{q}_0}\frac{\mathbf{p}_0'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_t}}$ → new quantity component

new price component $P^F = PC^*(SC^*)^{1/2}$	new quantity component $Q^F = QC^*(SC^*)^{1/2}$
$P_{0t}^F = \sqrt{P_{0t}^L P_{0t}^P} = \sqrt{\frac{\mathbf{p}_t'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_0}\frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_0'\mathbf{q}_t}}$	$Q_{0t}^F = \sqrt{Q_{0t}^L Q_{0t}^P} = \sqrt{\frac{\mathbf{q}_t'\mathbf{p}_0}{\mathbf{q}_0'\mathbf{p}_0}\frac{\mathbf{q}_t'\mathbf{p}_t}{\mathbf{q}_0'\mathbf{p}_t}}$

(3.6.4a) $\qquad \sum \ln(v_{it} / v_{i0}) = \sum \ln(p_{it} / p_{i0}) + \sum \ln(q_{it} / q_{i0})$.

The key equations explaining why P_{0t}^{V1} and Q_{0t}^{V1}, or P_{0t}^{V2} and Q_{0t}^{V2} are satisfying the factor reversal test and also relating Vartia-I- and Vartia-II weights are

(3.6.5) $\ln(v_{i0}) = \ln(w_{i0}) + \ln(V_0)$ and correspondingly $\ln(v_{it}) = \ln(w_{it}) + \ln(V_t)$

(3.6.6) $$\frac{L(v_{it}, v_{i0})}{L(V_t, V_0)} \ln(v_{it} / v_{i0}) = \frac{v_{it} - v_{i0}}{V_t - V_0} \ln(V_{0t})$$
$$= \frac{w_{it} - w_{i0}}{\ln w_{it} - \ln w_{i0}} \ln(v_{it} / v_{i0}) / \sum L(w_{it}, w_{i0})$$
$$= [(w_{it} - w_{i0}) + \ln(V_{0t}) L(w_{it}, w_{i0})] / \sum L(w_{it}, w_{i0})$$

Figure 3.6.3: Vartia's way of deriving ideal log-change-indices

a) Micromodel in log changes

(3.6.4) $\ln \frac{v_{it}}{v_{i0}} = \ln \frac{p_{it}}{p_{i0}} + \ln \frac{q_{it}}{q_{i0}}$ since by definition $\frac{v_{it}}{v_{i0}} = \frac{p_{it}}{p_{i0}} \frac{q_{it}}{q_{i0}} =$

b) Aggregation problem

$\sum \left(\ln \frac{v_{it}}{v_{i0}} \right) \neq \ln(V_{0t})$ find weights g such that

$\sum_i \left(g_i \cdot \ln \frac{v_{it}}{v_{i0}} \right) = \sum_i \left(g_i \cdot \ln \frac{p_{it}}{p_{i0}} \right) + \sum_i \left(g_i \cdot \ln \frac{q_{it}}{q_{i0}} \right)$ $\Rightarrow$ two solutions (Vartia I and II)

Figure 3.6.4: Weights in aggregating log-changes, Vartia's solution[1]

Single commodity (i) equation (definition): (3.6.4) $\ln(v_{it} / v_{i0}) = \ln(p_{it} / p_{i0}) + \ln(q_{it} / q_{i0})$

Aggregation: $\sum g_i \ln(v_{it} / v_{i0}) = \sum g_i \ln(p_{it} / p_{i0}) + \sum g_i \ln(q_{it} / q_{i0})$

Vartia I weights	Vartia II weights
$g_i = \frac{L(v_{i0}, v_{it})}{L(V_0, V_t)}$, $V_0 = \sum v_{i0}$, $V_t = \sum v_{it}$.	$g_i = \frac{L(w_{it}, w_{i0})}{\sum L(w_{it}, w_{i0})}$
$\sum g_i \ln(v_{it} / v_{i0}) = \ln(V_{0t}) \sum \left(\frac{v_{it} - v_{i0}}{V_t - V_0} \right)$ $= \ln(V_{0t})$, since $\sum \left(\frac{v_{it} - v_{i0}}{V_t - V_0} \right) = 1$	$\sum g_i \ln(v_{it} / v_{i0}) =$ $= \frac{\sum \left(\frac{w_{it} - w_{i0}}{\ln w_{it} - \ln w_{i0}} \ln(v_{it} / v_{i0}) \right)}{\sum L(w_{it} / w_{i0})} = \ln(V_{0t})$

1 The problem is *not* to show that $\sum g_i \ln(p_{it} / p_{i0})$ is $\ln(P_{0t}^{V1})$, or $\ln(P_{0t}^{V2})$ because these indices are *defined* this way. The same is true for quantity indices. It only remains to prove that the weighted aggregation $\sum g_i \ln(v_{it} / v_{i0})$ results in $\ln(V_{0t})$, with value index V_{0t}.

2 see eq. 3.6.6

Theil's Best Linear Index (BLI) – the two situation case (times 0, t) –

A pair of indices P^* and Q^* forming a matrix $\mathbf{D}^* = \begin{bmatrix} 1 & Q_{0t}^* \\ P_{0t}^* & P_{0t}^* Q_{0t}^* \end{bmatrix} = \mathbf{p} \cdot \mathbf{q}'$ is defined by minimizing the sum of squared differences of the elements of $\mathbf{D}^*$ and $\mathbf{D} = \begin{bmatrix} 1 & Q_{0t}^L \\ P_{0t}^L & V_{0t} \end{bmatrix}$ where V_{0t} in $\mathbf{D}$ is of course $V_{0t} = P_{0t}^L Q_{0t}^L (1 + C)$ and $\mathbf{p} = \begin{bmatrix} 1 \\ P_{0t}^* \end{bmatrix}$, $\mathbf{q}' = \begin{bmatrix} 1 & Q_{0t}^* \end{bmatrix}$.

Note: This is not the end of the Index-Theory part of the course. As all index formulas can be conceived as direct indices, or as chain indices (links), there will also be some more index theory presented in chapter 7. Moreover, as most index formulas have been recommended as deflators too, we also deal with index theory in chapter 5 (in particular with the aggregation issue). Finally formulas such as P^F or P^T have also been proposed as appropriate for international comparisons. Thus they will appear in chapter 8 once again.[19]

Chapter 4 Price collection, quality adjustment and sampling in official statistics

4.1. The set up of a system of price quotations and price indices in official statistics (the example of Germany)

Price quotations and compilations of indices have to be timely, regularly and systematic. They should consistently cover a wide field of market activities (sales/purchases), serve a great variety of users and purposes, and should be carried out by a *neutral*, competent and *trustworthy* institution[20]. In some instances it will be more, in other less difficult to arrive at correct prices and weighting schemes for price indices, and there is also often a need for compromises, sometimes even for makeshift solutions in the light of conflicting principles in price statistics.

In the implementation of a system of surveys in price statistics decisions have to be made on:

1. the kind of prices to be collected (***scope*** of price statistics),
2. the ***source*** of information best to be used (e.g. sellers or buyers[21]), and
3. the ***periodicity*** of regular surveys.

What **type of prices** should be observed: actual transaction prices vs. list prices; when a contract is made, the transaction effectively occurs, or when consumption takes place; excluding/including VAT.

The denomination of the index should denote

a) whether goods (and their prices), and weights refer to the supply side (sales) or the demand side (purchases),
b) the branch or sector (or the type of business involved) the index refers (institutional as opposed to functional approach).

Remark concerning price indices *plus* unit value indices (see sec. 6.4) in foreign trade statistics (a peculiarity of Germany) and some services (air transport for example)

- unit value indices display more (or too much) volatility and they are said to be less suitable for deflation than true price indices;
- price indices are true Laspeyres indices, reflecting pure price movement, and they refer to prices at an early stage (when a contract is made) and to narrowly defined commodity groups, so they may have a lead relative to unit value indices.

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[19] PF and PT are of interest because they comply with "country reversibility" (the interspatial criterion of what is known as time reversibility in the intertemporal context).

[20] Official statistics has to avoid the impression of applying questionable methods, or of experimenting with concepts and formulas that are difficult to understand and sometimes advanced not without some political interest.

[21] There are very few cases in which buyers can give adequate and competent information on prices on a regular basis, taking into account also all of the price determining characteristics (PDCs, like quality for example) and the changes in the PDCs. On the other hand it has sometimes been claimed that in a democratic system consumer prices should be reported by the many buying households and not by the few selling enterprises.