b) Uses and limitations: PPP and exchange rates

Figure 8.1.2: Uses of international comparisons of prices

International comparisons of prices

purchasing power parities (PPP)	deflating aggregates
the purpose is to estimate • an equivalent remuneration of employees when working abroad; • gains or losses of purchasing power (compared with the exchange rate); • PPP as indicators of performance in fighting inflation and international competition.	Task within the framework of NA • to make comparisons of level and structure of aggregates* • to aggregate volumes across countries (taking into account the different size of countries)**, as e.g. the European Union

* hence consistency in aggregation requirements should be obeyed more strictly than in the case of PPPs

** in the case of PPPs taking size of countries or of transaction - quantities involved is in general not needed.

c) Bilateral and multilateral (differences)

d) Some methods of bilateral and multilateral comparison

Figure 8.1.3: Comparisons in intertemporal and interspatial case direct comparisons

	four points in time			
	0	1	2	3
0	1	P_{01}	P_{02}	P_{03}
1	P_{10}	1	P_{12}	P_{13}
2	P_{20}	P_{10}	1	P_{23}
3	P_{30}	P_{31}	P_{32}	1

	four countries			
	A	B	C	D
A	1	P_{AB}	P_{AC}	P_{AD}
B	P_{BA}	1	P_{BC}	P_{BD}
C	P_{CA}	P_{CB}	1	P_{CD}
D	P_{DA}	P_{DB}	P_{DC}	1

Indirect comparisons (via one or two "third" countries)

pair	0 (direct)	1	2
A-B	A-B	A-C-B, A-D-B	A-C-D-B, A-D-C-B
A-C	A-C	A-B-C, A-D-C	A-B-D-C, A-D-B-C
A-D	A-D	A-B-D, A-C-D	A-B-C-D, A-C-B-D
B-C	B-C	B-A-C, B-D-C	B-A-D-C, B-D-A-C
B-D	B-D	B-A-D, B-C-D	B-A-C-D, B-C-A-D
C-D	C-D	C-A-D, C-B-D	C-A-B-D, C-B-A-D
sum	6	12	12

In addition there are also numerous indirect comparisons between any two fixed countries, say A and B, that is for each pair there exist m - 2 comparisons between two countries via one third country, (m-2)(m-3) comparisons via two "third countries", and (m-2)(m-3)(m-4) comparisons via three third countries and so on.

Hence in the case of 4 countries we have (see **fig. 8.1.3**) 6 *direct* comparisons (shaded) *plus*

- m - 2 = 2 indirect comparisons via *one* third country for each of the six pairs, like A-C-B and A-D-B in the case of the pair A-B
- (m - 2)(m - 3) = 2 indirect comparisons via *two* third countries for each of the six pairs, like A-C-D-B and A-D-C-B,

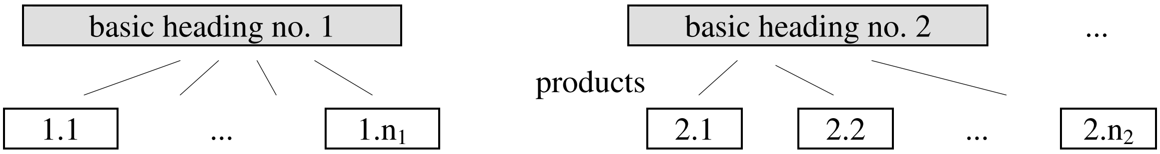
thus altogether 6 + 12+12 = 30 which have to be consistent with one another Correspondingly in the case of m = 5 countries the number of direct and indirect comparisons between two countries, that have to be consistent with one another grows up to 160, and with m = 6 already to no less than 15·65 = 975 reasonable comparisons..

Figure 8.1.4: Usage of notions, like Laspeyres and Paasche

	1. Unweighted indices (parities)	2. (weighted) price indices
Principle	the country from which the list of commodities is taken	the country from which this list and the weights (expenditure shares) are taken
Laspeyres	(8.1.1) $\quad {}_A L_B = \left(\prod_{i=1}^{i=n_A} \frac{p_{Bi}}{p_{Ai}}\right)^{1/n_A}$	(8.1.2) $\quad P_{AB}^{L} = \frac{\Sigma p_B q_A}{\Sigma p_A q_A}$
Paasche	(8.1.1a) $\quad {}_A P_B = \left(\prod_{k=1}^{n_B} \frac{p_{Bk}}{p_{Ak}}\right)^{1/n_B}$	(8.1.2a) $\quad P_{AB}^{P} = \frac{\sum p_B q_B}{\sum p_A q_B}$

A = base country, B = reference country, P_A and P_B absolute prices in A and B; commodities i = 1,..., n_A preferred by country A; k =1,..., n_B = preferred by country B

Products and "basic headings" in case 1



e) Consistency in multinational comparisons (the meaning of transitivity)

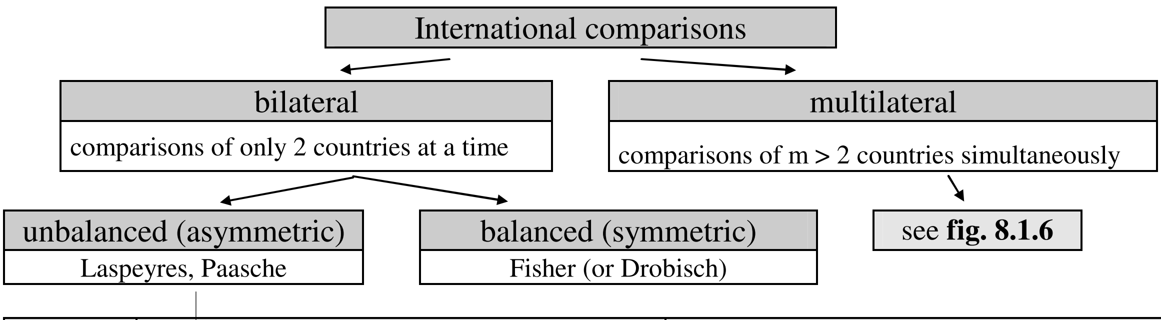
Fisher parities are *not transitive*

$$\hat{P}_{AB(C)}^{F} = \frac{P_{CB}^{F}}{P_{CA}^{F}} = P_{AC}^{F} P_{CB}^{F} = \sqrt{\frac{(\mathbf{p}_C'\mathbf{q}_A)\cdot(\mathbf{p}_B'\mathbf{q}_C)\cdot(\mathbf{p}_B'\mathbf{q}_B)}{(\mathbf{p}_A'\mathbf{q}_A)\cdot(\mathbf{p}_A'\mathbf{q}_C)\cdot(\mathbf{p}_C'\mathbf{q}_B)}} = \sqrt{V_{AB}}\sqrt{\frac{(\mathbf{p}_C'\mathbf{q}_A)\cdot(\mathbf{p}_B'\mathbf{q}_C)}{(\mathbf{p}_A'\mathbf{q}_C)\cdot(\mathbf{p}_C'\mathbf{q}_B)}}$$

$$\hat{P}_{AB(D)}^{F} = \frac{P_{DB}^{F}}{P_{DA}^{F}} = P_{AD}^{F} P_{DB}^{F} = \sqrt{\frac{(\mathbf{p}_D'\mathbf{q}_A)\cdot(\mathbf{p}_B'\mathbf{q}_D)\cdot(\mathbf{p}_B'\mathbf{q}_B)}{(\mathbf{p}_A'\mathbf{q}_A)\cdot(\mathbf{p}_A'\mathbf{q}_D)\cdot(\mathbf{p}_D'\mathbf{q}_B)}} = \sqrt{V_{AB}}\sqrt{\frac{(\mathbf{p}_D'\mathbf{q}_A)\cdot(\mathbf{p}_B'\mathbf{q}_D)}{(\mathbf{p}_A'\mathbf{q}_D)\cdot(\mathbf{p}_D'\mathbf{q}_B)}}$$

They will in general differ and also differ from $P_{AB}^{F} = \sqrt{V_{AB}}\sqrt{\mathbf{p}_B'\mathbf{q}_A / \mathbf{p}_A'\mathbf{q}_B}$.

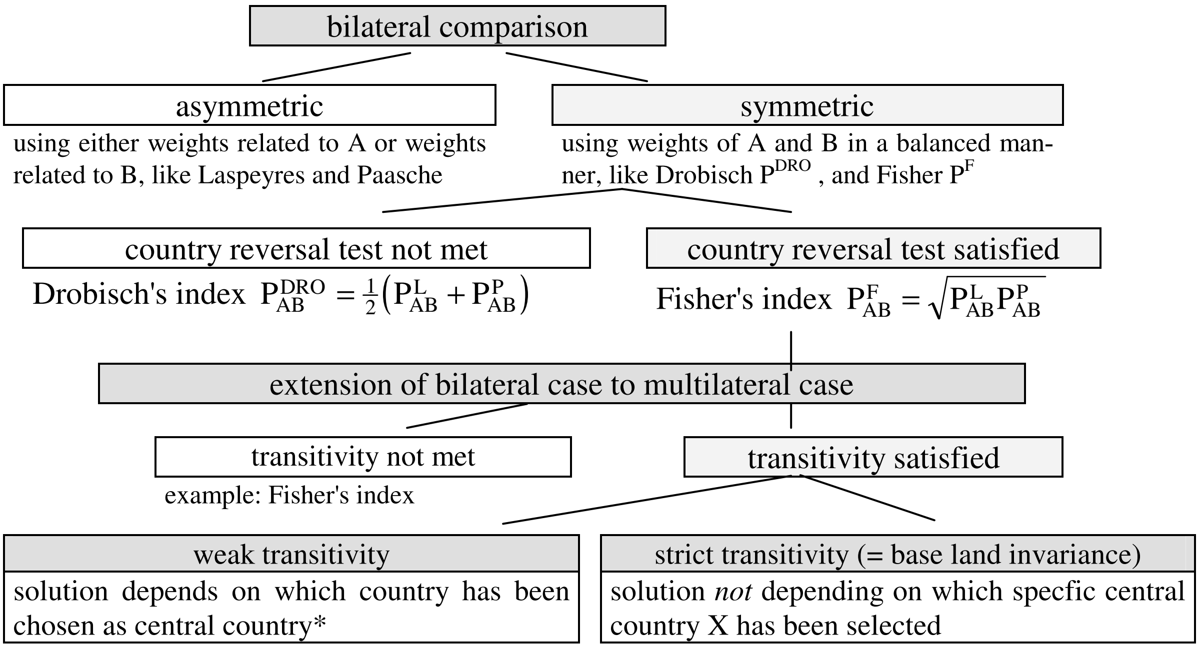
Fig 8.1.5: Bilateral and multilateral comparisons
Laspeyres- and Paasche- approach in bilateral international comparisons[1)]



	Laspeyres approach	Paasche approach
Interpre-tation	Base country budget q_A. Tourist model: a person P of country A visits B as tourist and is maintaining his consumption habits	Reference country (B) budget q_B. Migrator model; assimilation to the new conditions (consumption habits
Commodities	representative of country A	representative of country B
Country reversibility*	(8.1.2) $\quad P_{BA}^{L} = 1/P_{AB}^{P}$	(8.1.2a) $\quad P_{BA}^{P} = 1/P_{AB}^{L}$
Deviation from reciprocal index**	$D = P_{BA}^{L} / (P_{AB}^{L})^{-1} = P_{BA}^{L} P_{AB}^{L}$	$D^* = P_{BA}^{P} / (P_{AB}^{P})^{-1} = P_{BA}^{P} P_{AB}^{P} = 1/D$

*) Formulas 8.1.2 and 2a see also **fig. 8.1.4**
*) given (summation of an) identical commodity lists

Figure 8.1.6: From properties of bi-lateral comparisons (between place A and place B) to desired properties of multi-lateral comparisons



* example for this method: Central Country Method (CCM) making indirect comparisons between any two countries via a (the same in all comparisons) third "common" country X.

Transitivity (concerning multinational comparisons) is possible only if *all* indirect comparisons between any two countries, A and B that is P_{AB} (or Q_{AB} respectively) obtained by using other countries, like C as link should be equal to the direct index. None of the standard indices in common use (like Laspeyres, Paasche and Fisher) is able to ensure weak (let alone strict) transitivity.

f) Conditions, axioms and required properties in multinational comparisons

Characteristicity requires that commodities and quantity weights are used so that

- not only all countries to be compared are represented, but also that
- commodities and weights provide an adequate coverage and representation of the consumption in the different countries under consideration.

Diewert's set of tests for multinational comparisons (focused on volume comparisons)

D1 positivity and continuity (in all arguments) of volume shares Q_i (i = 1, ..,m countries),
D2 symmetric treatment of countries (i.e. invariance of vol. shares to permutation of countries),
D3 symmetric treatment of commodities test,
D4 monetary unit test (= invariance to changes in scale test) corresponds to price dimensionality in the intertemporal case: replacing the vector $\mathbf{p}_i$ by $\alpha_i\mathbf{p}_i$ and $\mathbf{q}_i$ by $\beta\mathbf{q}_i$ should not affect the volume share of country i.[70]
D5 invariance to changes in units of measurement (plays the part of the commensurability),
D6 country partitioning test: let country h be partitioned into part (province) 1 with a quantity vector $\lambda\mathbf{q}_h$ and a part 2 with $(1-\lambda)\mathbf{q}_h$, and the same price vector for both parts, then the quantity shares should be λQ_h and $(1-\lambda) Q_h$ (or: small countries should not influence the volume shares of large countries unduly),
D7 irrelevance of tiny countries test: denote $\lambda\mathbf{q}_k$ the quantity vector of a (tiny) country k; if $\lambda \to 0$ quantity shares of all countries should tend to the quantity shares we get if calculations are done exclusive of country k,

[70] different inflation rates α_i but equal quantity growth rates β leave the quantity shares unchanged.

D8 (*weak*) *proportionality* (and hence also identity) test w.r.t. prices (or equivalently w. r. t. quantities): upon substitution of $\mathbf{p}_i$ by $\alpha_i \mathbf{p}_i$ and quantities in the same manner, i.e. substitution of $\mathbf{q}_i$ by $\beta_i \mathbf{q}_i$ - that is equality of the *structure* (*not* the *level*) of prices or quantities across all m countries - should result in quantity shares dependent on β_i only (= D8 "w.r.t. quantities")

D9 *proportionality* test: replace for country h the vector $\mathbf{q}_h$ by $\lambda \mathbf{q}_h$ and the scalar (non-normalized) country weight g_h by λg_h then the quantity share of this country h should change from Q_h to $\lambda Q_h/[1+(\lambda-1]Q_h$ and the share of any other country i ≠ h should change accordingly (effect of a changes in the mere size of a country h expressed in a uniform λ-fold change in all its quantities). In particular tests D9 on the one hand and D6/D7 on the other seem to be inconclusive.

Figure 8.1.7: Criteria and requirements for international comparisons

This list should not be misinterpreted as a set of non-contradictory (consistent) and independent axioms as for example the axiomatic system of Eichhorn and Voeller (see **sec. 3.3**).

Axiom	Meaning	Remark
1 Time (country) reversal test	Unique parity among two countries, A and B, such that $P_{AB} = 1/P_{BA}$ (choice of base country irrelevant)[1].	Not to be reconciled with no. 2, at least not in the case of widely differing countries [2].
2 Characteristicity (typicality or equidistance)	Indices should be representative for both (all) countries to be compared with respect to kind and quantities of commodities [3].	Condition referring to type of weights (unusual in axiomatic theory)
3 Unbiasedness **4** Mean value (or: average) test (**4a** for prices **4b** for quantities)	A parity (price index) should lie within the interval between P^P and P^L, it should also meet the mean value condition {= average test 4a} (the same is desired for quantity indices [4], = test 4b {independent of 4a}).	Relevance and meaning of no. 4 is difficult to distinguish from no. 3; not clear why no. 4 this should apply to PPPs too.
5a Additivity (I) (or: **structural** consistency [of *volumes*])	Structural consistency in the sense of **sec. 5.2**, useful to make consistent National Accounts for a block of countries like the EEC.	Required when deflators or volume indices are used in an accounting framework.
5b Additivity (II) (or: **aggregative** consistency [of the index formula] see **sec. 5.2**)	Required in particular for quantity indices Q, such that the overall Q can easily be decomposed in sub-indices measuring the quantity movement of sub-aggregates	Useful if comparisons are made at varying levels of aggregation [5] (as for example in National Accounts)
6a Weak Factor Reversal Test (WFR)	Price (P) and quantity index (Q) are related to the value index (V) by PQ = V; also known as **product test**	Allows for a consistent breakdown of volumes into a P - and Q - component
6b Strict (Strong) **Factor Rev. Test** (SFR)	Like WFR and in addition: P can be obtained from Q by interchanging prices and quantities (likewise Q from P)	Implies that the method to derive P (or Q respectively) is symmetric (balanced)
7 Transitivity **7a** weak, **7b** strict transitivity)	The direct comparison and indirect comparisons of all [6] (strict transitivity) sorts should yield the same (identical) result.	More important than in inter-temporal comparisons, but difficult to achieve.

1) Not only the country but also time reversal test has been criticized: History cannot be made undone, there is no "run backward" and more often than not no meaningful result to expect (it is absurd to ask for the price of a flight-ticket 1890).

2) It is argued that the result cannot be trustworthy exactly *because* the country reversal test is satisfied. This is particularly convincing in the case of very different countries (e.g. Germany and India).

3) If any two countries are treated symmetrically with respect to quantities q_A, q_B as in the case of Fisher (or Drobisch) indices the approach is *equidistant*. Equidistant indices usually are satisfying the product- or even the factor reversal-test.

4) This is supposed to be a necessary condition to perform real-value comparisons between various countries on different levels of aggregation.

5) Requires a method using a single vector of prices (like the vector of average prices of the community in the Geary Khamis method)

6) Known as strict transitivity, or *base-land-invariance* (a criterion that should be kept distinct from country reversibility. It is possible that transitivity is given only by indirectly comparing with a *specified unique* "third" country (for example the "central" or "star" country). This is known as *weak transitivity*.

8.2. Overview of methods proposed for multinational comparisons

a) Introduction into solutions of transitivity	d) Method of minimum spanning trees (MST)
b) Evaluation of methods (adequate for EU)	e) Comments on other methods
c) Block methods (Geary Khamis {GK})	

a) Methods to solve the transitivity problem

Fig. 8.2.1 is an attempt to find a structure for the multinational methods, however, some methods as for example methods proposed by van Yzeren, or the *Minimum Spanning Tree (MST) Method* cannot adequately be accounted for. Block methods can be described as follows

1. to derive transitive inter-country comparisons of m countries with respect to price indices P_{AB} or P_{ij} (quantity indices Q_{ij} correspondingly) we have to define m positive real numbers λ_1, λ_2, ..., λ_m, such that $P_{ij} = \lambda_i \lambda_j$ or more convenient $P_{ij} = P_i/P_j$ and $Q_{ij} = Q_i/Q_j$
2. In order to comply with the product test the following equation should hold

(8.2.1) $$V_{kj} = \Sigma p_j q_j / \Sigma p_k q_k = P_{kj} Q_{kj} = (P_j/P_k)(Q_j/Q_k),\ k, j = 1, ..., m.$$

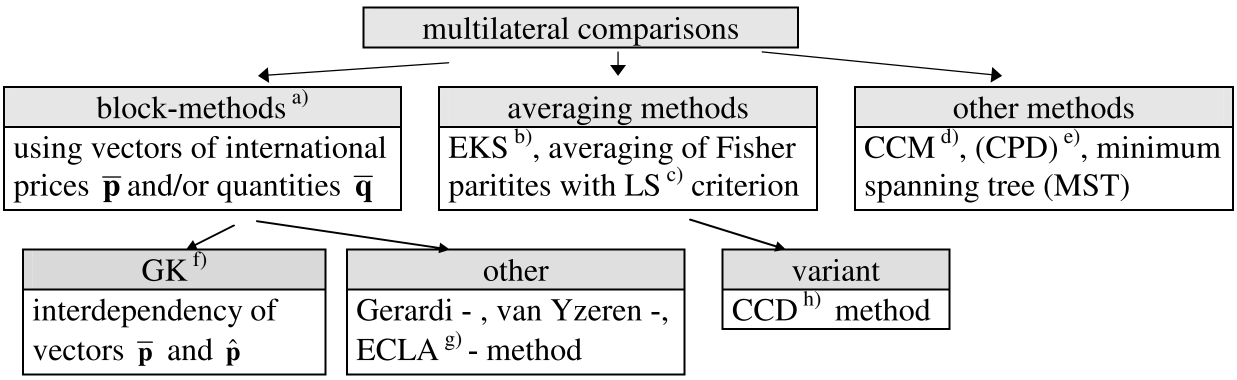
The methods listed in **fig. 8.2.1** can be distinguished depending on how P_{kj} and Q_{kj} are defined, viz.

- by either referring to an average (artificial, central, block) country, as for example in the case of the GK method, or
- by averaging over all binary comparisons as regards prices or quantities respectively of the m countries to be compared (EKS- and related methods[71]).

Figure 8.2.1: Overview of most relevant methods, part I (esp. GK and EKS – Method)

Assume m (i = 1, ... , m) countries forming a block and n commodities (k = 1, ..., n), and

m vectors of the type	one (for the community) vector of
price vectors, $\mathbf{p}'_i = [p_{1i} \ldots p_{ni}]$ for country i with n prices expressed in its own (the i-th country) currency	international prices $\bar{\mathbf{p}}' = [\bar{p}_1 \ldots \bar{p}_n]$ expressed in the block's currency and if necessary also of international quantities $\bar{\mathbf{q}}'$
quantity vectors $\mathbf{q}'_i = [q_{1i} \ldots q_{ni}]$ of n quantities in country i	parities $\hat{\mathbf{p}}' = [P_1 \ldots P_m]$ or vector of volumes $\hat{\mathbf{q}}'$ as a result of the method



a) methods treating the block as an entity of its own

b) Eltetö - Köves - Szulc Method

c) least squares

d) Central Country Method

e)Country-Product-Dummy Method (a regression method)

f) Geary - Khamis - Method

g) Economic Commission for Latin America Method

h) Caves-Christensen-Diewert Method

[71] which therefore may also be called "generalizations of binary comparisons" (Balk).

Eq. 8.2.1 may be specialized as follows (X denotes the central- or block-country)

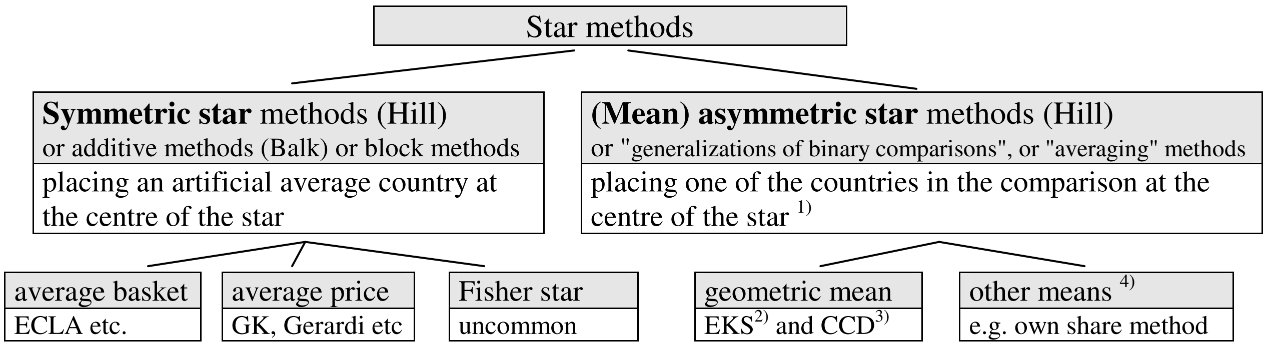
(8.2.2) $$\frac{P_j}{P_k}\frac{Q_j}{Q_k} = \frac{P^P_{xj}}{P^P_{xk}}\frac{Q^L_{xj}}{Q^L_{xk}} = \left(\frac{\sum p_j q_j}{\sum p_x q_j}\frac{\sum p_x q_k}{\sum p_k q_k}\right)\left(\frac{\sum p_x q_j}{\sum p_x q_k}\right) = \frac{\sum p_j q_j}{\sum p_k q_k} = V_{kj}$$

such that both factors P_j/P_k and Q_j/Q_k depend on prices p_x (not on quantities q_x) of the central country only (note that this is true for the second factor $Q^L_{xj}\big/Q^L_{xk} = \sum p_x q_j \big/ \sum p_x q_k$). We thus may rightly call methods on this basis *average price methods* (**fig. 8.2.2**), of which the GK method (Geary - Khamis) is an example. In a similar vein eq. 8.2.1 may be specialised as

(8.2.3) $$\frac{P_j}{P_k}\frac{Q_j}{Q_k} = \frac{P^L_{xj}}{P^L_{xk}}\frac{Q^P_{xj}}{Q^P_{xk}} = \left(\frac{\sum p_j q_x}{\sum p_k q_x}\right)\left(\frac{\sum p_j q_j}{\sum p_j q_x}\frac{\sum p_k q_x}{\sum p_k q_k}\right)$$

where both factors depend on the x-country's quantities only, such that methods based on eq. 8.2.3 (as eg the ECLA method) may be called *average basket methods* making use of a vector of common quantities $\mathbf{q}_x$ or $\bar{\mathbf{q}}' = [\bar{q}_1 ... \bar{q}_n]$ rather than of prices. For practical reasons this approach has much less to commend it than the *average price methods*.

Figure 8.2.2: Overview of some of the most relevant methods, part II
(with reference to R. J. Hill 1997 and B. M. Balk 2001)*



*) The scheme is again not exhaustive in that some methods like eg the minimum spanning tree, regression and Multivariate Generalized Törnquist (MGT) method are not given mention. Furthermore van Yzeren proposed methods belong to different categories in this system. According to Hill most methods can be regarded as one or another variant of star methods (i.e. making indirect comparisons over a third country, the centre of a star X)

1) this description would of course also fit to the Central Country Method (CCM) 2) using Fisher indices
3) Caves-Christensen-Diewert Method using Törnquist indices
4) much less common variants using e.g. the arithmetic or harmonic mean

(8.2.4) $$\frac{P_j}{P_k}\frac{Q_j}{Q_k} = \frac{P^F_{xj}}{P^F_{xk}}\frac{Q^F_{xj}}{Q^F_{xk}}$$

giving rise to the much less known "Fisher star method" listed in **fig. 8.2.2**. Methods which may be viewed as "generalizations of binary comparisons" (Balk) conceptualise ratios like P_j/P_k (and in a similar manner Q_j/Q_k) as follows[72]

(8.2.5) $$\frac{P_j}{P_k} = \underset{i}{M}\left(\frac{P^X_{ij}}{P^X_{ik}}\right) = \underset{i}{M}\left(P^X_{ij}P^X_{ki}\right)$$

where $\underset{i}{M}(...)$ denotes a mean of binary comparisons (over all i = 1, ..., m countries) using index formulas of type X. For example the well known formula in the EKS method

(8.2.5a) $$P^{EKS}_{AC} = \left[\left(P^F_{AC}P^F_{AA}\right)\left(P^F_{BC}P^F_{AB}\right)\left(P^F_{CC}P^F_{AC}\right)\right]^{1/3} = \sqrt[3]{\left(P^F_{AC}\right)^2 P^F_{AB}P^F_{BC}}$$

[72] This applies to the EKS method or variants of it, as for example the CCD-method using P^T instead of P^F.

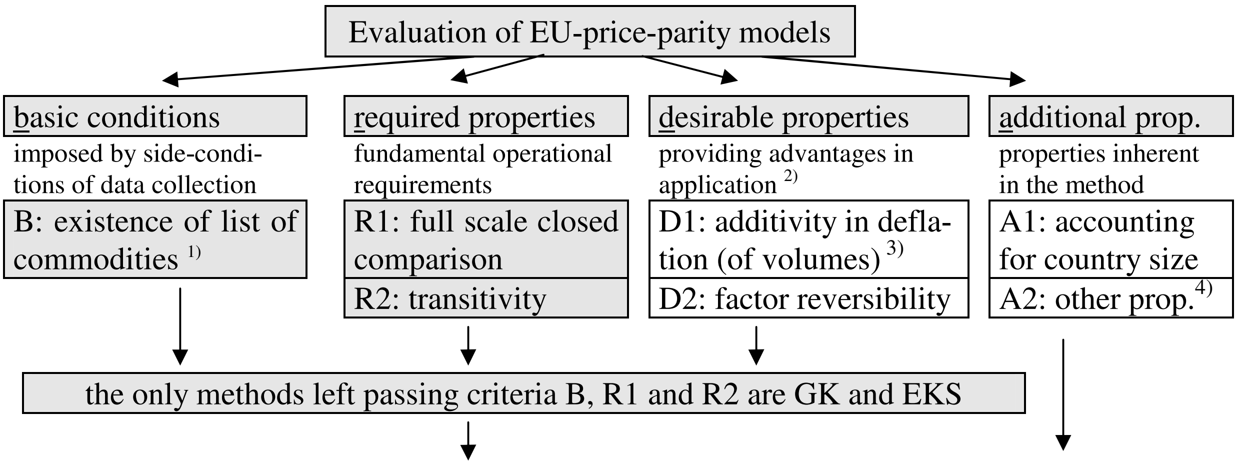
represents a geometric mean of bilateral comparisons on the basis of Fisher (X = F) indices.

(8.2.5b) $\frac{P_j}{P_k} = \left(\frac{P^F_{1j}}{P^F_{1k}} \frac{P^F_{2j}}{P^F_{2k}} \cdots \frac{P^F_{mj}}{P^F_{mk}} \right)^{1/m}$ and using $P^F_{BA} = 1/P^F_{AB}$ we get

$$\left(\frac{P^F_{AC}}{P^F_{AA}} \frac{P^F_{BC}}{P^F_{BA}} \frac{P^F_{CC}}{P^F_{AC}} \right)^{1/3} = \sqrt[3]{\left(P^F_{AC}\right)^2 P^F_{AB} P^F_{BC}}$$ in the three - countries example.

b) Evaluation of methods adequate for intra-EU comparisons

Figure 8.2.3: Criteria to find suitable methods for inter-EU-comparisons



method	D1: additivity	D2: factor reversal test	A1: country size [5]
GK	not satisfied (quasi-additivity in comparisons of a country i to the block as a whole)	violated (only product test), price and quantity parities differently constructed	size of country i affects results via quantities q_{ki} [7]
EKS	violated [6]	satisfied [6]	no influence

1) incomplete list of commodities can be handled
2) also theoretical elegance 3) = structural consistency in the sense of **sec. 5.2**
4) meaningful parameters provided as by-product of the method, i.e. methods permits interesting interpretations
5) or "importance" of a country 6) both results due to EKS parities being based on *geometric* means
7) however counter-intuitively

c) Block methods: the Geary-Khamis (GK) method

The key idea of the GK-method is to determine the "international" prices $\bar{p}_k$ of commodities (k = 1, 2, ..., n) and m currency converters c_i (of country i =1, 2, ..., m) *simultaneously*. The common (community-, or block-) price $\bar{p}_k$ of commodity k is defined with the help of the m "currency converter" c_i of country i as follows (a system of n equations, the summation takes place over m countries):

(8.2.6) $$\bar{p}_k = \frac{\Sigma c_i p_{ki} q_{ki}}{\Sigma q_{ki}} = \frac{\Sigma c_i p_{ki} q_{ki}}{Q_k} = \sum_{i=1}^{m} c_i p_{ki} \alpha_{ki}\text{, where } \alpha_{ki} = q_{ki}/Q_k \text{ or } \bar{p}_k = \sum c_i \tilde{p}_{ki}$$

where $\tilde{p}_{ki}$ denote *unit values* and Q_k is a sum of quantities (over all m countries), α_{ki} is a structural variable accounting for the size of a country i, and c_j is, as aforementioned, a currency converter (reciprocal exchange rate)[73] in order to allow for expenditures in the numera-

[73] Unfortunately there is always some confusion because some writers use converters, c_i as above, others prefer to express the equations in terms of exchange rates $e_i = 1/c_i$.

tor expressed in the same common currency unit. The converters c_i define the price level [or parity] of country i with respect to the whole community as follows

(8.2.7) $c_i = \frac{\Sigma \bar{p}_k q_{ki}}{\Sigma p_{ki} q_{ki}} = \frac{\Sigma \bar{p}_k q_{ki}}{V_i}$ a system of m equations for m countries.

In eq. 8.2.7 summation takes place over k = 1, ... , n commodities, and V_i is the total value of country i). The system allows, however, only for calculation of m-1 coefficients c_i expressed in units of one of the c_i - coefficients, say c_2, c_3, … in units of c_i. This is sufficient as the aim is to define (purchasing power) parities between any two countries, A and B with reference to the community as a whole. The GK-parity between countries A, B now is defined as follows

(8.2.8) $P_{AB}^{GK} = \frac{c_A}{c_B} = \frac{e_B}{e_A} = \frac{\Sigma p_{kB} q_{kB} / \Sigma \bar{p}_k q_{kB}}{\Sigma p_{kA} q_{kA} / \Sigma \bar{p}_k q_{kA}}$.

This way of expressing GK-parity between any two countries, A (base) and B makes clear that

1. *identity* is given, that is when all m prices for the k-th commodity are equal $p_{k1} = ... p_{km} = \bar{p}_k$ then all parities will be unity, or $c_i = c_j = 1$;
2. (strict) *transitivity* as well as *country reversibility* holds since

   (8.2.8a) $P_{AC}^{GK} = P_{AB}^{GK} P_{BC}^{GK} = \left(\frac{c_A}{c_B}\right)\left(\frac{c_B}{c_C}\right)$,

   due to using a constant (for all countries) vector, $\bar{\mathbf{p}}'$ all indices, P_{AB}^{GK}, V_{AB}, and Q_{AB}^{GK} are transitive;
3. the *product test* (weak factor reversal test) is met since the factor antithesis of P_{AB}^{GK} is

   (8.2.8b) $Q_{AB}^{GK} = \frac{\Sigma \bar{p}_k q_{kB}}{\Sigma \bar{p}_k q_{kA}} = \frac{Q_B}{Q_A}$, but the (strict) *factor reversal* test is *not* satisfied, because the quantity index gained from P_{AB}^{GK} by interchanging prices and quantities would be $Q_{AB}^{*} = \frac{\Sigma \bar{q}_k p_{kA}}{\Sigma \bar{q}_k p_{kB}} \frac{\Sigma p_{kB} q_{kB}}{\Sigma p_{kA} q_{kA}} \neq Q_{AB}^{GK} = \frac{\Sigma \bar{p}_k q_{kB}}{\Sigma \bar{p}_k q_{kA}}$.
4. Both indices, P_{AB}^{GK} and Q_{AB}^{GK} are "additive" index functions:

$V_A = \Sigma p_{kA} q_{kA}$	$Q_A = \Sigma \bar{p}_k q_{kA}$	$c_A = Q_A/V_A$
$V_B = \Sigma p_{kB} q_{kB}$	$Q_B = \Sigma \bar{p}_k q_{kB}$	$c_B = Q_B/V_B$
$V_{AB} = \frac{V_B}{V_A}$	$Q_{AB}^{GK} = \frac{Q_B}{Q_A}$	$P_{AB}^{GK} = \frac{c_A}{c_B}$

   By virtue of these relationships all GK - indices, P^{GK} and Q^{GK} can be broken down to the commodity level and aggregated to whichever subindex is wanted. This does *not*, however, imply that structural consistency of volumes in deflation is given.
5. What is responsible for c_A, c_B etc., and hence also for P_{AB}^{GK} is the extent to which the prices in A, or B respectively differ from the prices valid for the community of all m countries, or in other words, what matters is the extent to which
   - *values* (quantities of country i expressed in prices in their *own* currency) differ from
   - *volumes* (that is "deflated" values $\Sigma \bar{p}_k q_{ki} = Q_i$, or quantities of country i valued at *common* prices).
6. The GK-method tends to a price index P^{GK} which is dominated by the small (unimportant,) country while the quantity index Q^{GK} is likely to be dominated by the big (important) country.

In Gerardi's method the international price $\overline{p}_k$ of commodity k (in which c_i is expressed in the currency unit of the community) is an unweighted geometric mean (no country weights)

$$(8.2.9) \qquad \breve{p}_k = \left(\prod_{i=1}^{m} c_i p_{ki} \right)^{1/m} .$$

d) The method of "minimum spanning trees" (MST-Method)

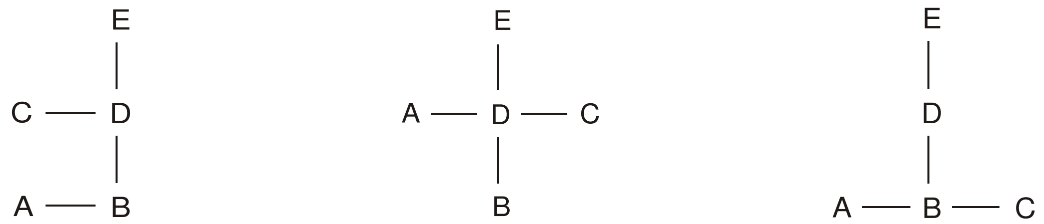
The basis of the method is the notion of a "distance" (dissimilarity) between any two countries as for example A and B using the Laspeyres-Paasche Spread or "Paasche-Laspeyres-Spread" (PLS) defined as

$$(8.2.10) \qquad D(A,B) = \left| \ln\left(\frac{P_{AB}^{L}}{P_{AB}^{P}} \right) \right| = \left| \ln\left(\frac{P_{AB}^{P}}{P_{AB}^{L}} \right) \right| .$$

If two countries are quite similar with respect to the structure of consumption the results of a P^L and a P^P-type index would not differ much such that D comes close to ln(1) = 0.

A spanning tree is a connection between a country (point, vertix, edge) i and each of the remaining m - 1 countries such that each country is (indirectly) linked with each other country in one way only. The "star" (second example in fig. 8.2.4) is a special spanning tree just like the chain (= "string") A-B-C-D-E. There are no two or more paths connecting any two countries (a situation which would be called "cycle"). **Fig. 8.2.4** gives some examples of spanning trees for m = 5 countries s along with fictitious numerical values of the distances[74] (ranging from 0.07 to 0.12) of the matrix of distances.

Figure 8.2.4: The notion of a "spanning tree"



-	0.08			
	-		0.07	
		-	0.10	
			-	0.12

sum: 0.37

-			0.10	
	-		0.07	
		-	0.10	
			-	0.12
				-

0.39

-	0.08			
	-	0.11	0.07	
		-		
			-	0.12
				-

0.38

The criterion for the *minimum* spanning tree (MST) is the sum of the distances which is in the examples of fig. 8.2.4 amounting to 0.37, 0.39 and 0.38. The minimum is obviously 0.37 such that the left configuration is the MST for the example[75].

The criterion of the smallest summed m-1 distances is equivalent to other reasonable criteria and it amounts to the distance of the chained spreads because

$$D(A,B) + D(B,C) = \left| \ln\left(\frac{P_{AB}^{L}}{P_{AB}^{P}} \frac{P_{BC}^{L}}{P_{BC}^{P}} \right) \right| \neq D(A,C) = \left| \ln\left(\frac{P_{AC}^{L}}{P_{AC}^{P}} \right) \right|$$

Transitivity would hold if and only if P^L and P^P indices were transitive

[74] In the left case we have for example D(A,B) = 0.08, D(B,D) = 0.07 etc.

[75] The total distance in the case of the above mentioned string A-B-C-D-E for example amounts to 0.08 + 0.11 + 0.10 + 0.12 = 0.41, and is thus much greater than 0.37.

e) Other methods

1. Regression (= *Country-Product-Dummy (CPD) Method)*, 2. Model based (COLI-type), 3. Multilateral generalized Törnquist method (MGT-index). With $v_{ik} = p_{ik}q_{ik}/\Sigma p_{ik}q_{ik}$ the share of total expenditure in country I spent n commodity k, the MGT quantity index is given as

(8.2.12) $$Q_{ij}^{MGT} = \prod_{k=1}^{n}\left(\frac{q_{jk}}{q_{ik}}\right)^{m_k}$$ and by analogy the MGT *price* index $$P_{ij}^{MGT} = \prod_{k=1}^{n}\left(p_{jk}/p_{ik}\right)^{m_k}$$

where m_k is (in contrast to the "usual" Törnquist index) a function of expenditure shares *of all m countries*, not only of just the two compared ones, viz. country i (base) and j. Thus for example Walsh-type generalized MGT-index has been proposed where a geometric mean of all m countries as regards the expenditure share of commodity k to be taken for m_k

$$m_k = \prod_{i=1}^{m}(v_{ik})^{1/m} \Big/ \sum_{k=1}^{n}\prod_{i=1}^{m}(v_{ik})^{1/m}.$$

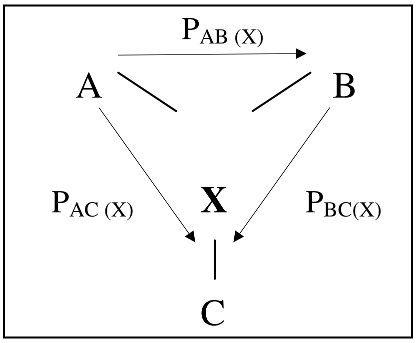
8.3. Block methods

a) Central Country Method (CCM)	c) Balanced method of van Yzeren
b) Geary Khamis (GK) method	d) Other block methods (Gerardi, ECLA)

a) Central Country Method (CCM)

(8.3.1) $$\hat{P}_{AB}^{L} = P_{AB(X)}^{L} = \frac{P_{XB}^{L}}{P_{XA}^{L}} = \frac{\sum p_B q_X}{\sum p_X q_X} \Big/ \frac{\sum p_A q_X}{\sum p_X q_X} = \frac{\sum p_B q_X}{\sum p_A q_X}.$$

Any two countries, A and B are compared only via X which therefore is also called the "star" country (or "bridge" or "link" country when the CCM is applied to two (or more) *groups* of countries). Note that the direction of the arrows is from base country to reference country. Thus $P_{BA(X)}$ instead of $P_{AB(X)}$ means to invert the direction of the arrow. CCM is a method easy to understand, however the results are not unique but depending on which choice has been made concerning the central country. Hence only weak transitivity is met.



(8.3.2) $$P_{BA(X)} = \left(P_{AB(X)}\right)^{-1}$$ (country reversibility holds)

for all countries, A and B, as well as the circular test

(8.3.3) $$P_{AC(X)}^{L} = P_{AB(X)}^{L}P_{BC(X)}^{L}$$ since $$\frac{\Sigma p_C q_X}{\Sigma p_A q_X} = \frac{\Sigma p_B q_X}{\Sigma p_A q_X}\frac{\Sigma p_C q_X}{\Sigma p_B q_X}.$$

Note that $P_{AB(X)}^{L}$ is still a Laspeyres type index, the identity of the base country (A) and the country from which the weights come (X) is destroyed, however. When Paasche parities are constructed in the same manner (by relating both, A and B to the third country X) we get more complicate expressions $P_{AB(X)}^{P} = \frac{P_{XB}^{P}}{P_{XA}^{P}} = \frac{\sum p_X q_A}{\sum p_A q_A}\frac{\sum p_B q_B}{\sum p_X q_B}$ etc.

CCM delivers non-characteristic results (**Tab. 8.3.1**): In multilateral comparisons the criterion of "characteristicity" or "specificity" is supposed to be desirable. This means that the weights q should be specific (typical) for the country in question (or the countries to be compared).

Table 8.3.1: Characteristicity (specificity) of the CCM solution*

Price index P			Quantity index Q
formula	quantities in P referring to	specificity	prices in Q referring to
Laspeyres $P^{L}_{AB(X)}$	X only	nonspecific	A and B
Paasche $P^{P}_{AB(X)}$	A and B	equi-specific	A, B and X
Fisher $P^{F}_{AB(X)}$	A, B and X	specific**	A, B and X

* all statements can easily be verified looking at eqs. 8.3.1, 4 and 5
** but three budgets involved, so that it is of particular interest that X is representative for all countries

b) Geary Khamis Method

The general approach of the Geary-Khamis method consists in simultaneously determining a price level of the "community" (or "block" of countries) as a whole (vector $\bar{\mathbf{p}}'$ of n prices for n commodities) and a vector of parities $\hat{\mathbf{p}}' = [\, P^{GK}_{12} P^{GK}_{13} \ldots]$ with respect to a numeraire country as for example i = 1.

Figure 8.3.2: The main relationships in the GK-method

determine simultaneously using national prices and quantities p_{ki}, q_{ki}

n community prices (k = 1, ..., n)

(8.2.6) $\bar{p}_k = \sum_{i=1}^{m} c_i p_{ki} \frac{q_{ki}}{\sum_i q_{ki}} = \sum_{i=1}^{m} c_i p_{ki} \alpha_{ki}$

n equations, functions of national prices p_{ki} and national quantities q_{ki}

m currency converters (i = 1, ..., m)

(8.2.7) $c_i = \sum_{k=1}^{n} \bar{p}_k q_{ki} \Big/ \sum_{k=1}^{n} p_{ki} q_{ki} = Q_i / V_i$

m equations, relating national price levels to price level of the community

m quantity indices

(8.2.8b) $Q^{GK}_{ij} = \frac{\sum \bar{p}_k q_{kj}}{\sum \bar{p}_k q_{ki}} = \frac{Q_j}{Q_i}$

m price indices

(8.2.8) $P^{GK}_{ij} = \frac{c_i}{c_j} = \frac{\sum \bar{p}_k q_{ki} / \sum p_{ki} q_{ki}}{\sum \bar{p}_k q_{kj} / \sum p_{kj} q_{kj}}$

The major shortcomings of the GK-method are

- the factor reversal test is violated, the product test satisfied, however (see **sec. 8.2**),
- equi-*characteristicity* is violated; interestingly the result is in fact more characteristic for a peripherical than for a central country,
- as to *consistency in aggregation* (criterion 4b in tab. 8.1.7) is violated while *structural consistency* (of volumes) is satisfied by virtue of using common (community) prices for all countries,
- the GK-index system also violates the *price dimensionality* axiom in the sense of a uniform (λ-fold) change of some, not necessarily all prices,
- finally an argument of considerable practical importance concerning data requirements: the GK method assumes data on *individual* prices and quantities (referring to *each of the n identically defined commodities* in the m countries) whilst in practice very often only binary-comparison data on the level of *index* numbers (comprising all n quantities) are available[76].

[76] For instance Fisher indices for all possible pairs of countries (full scale data) or for only some of the country-pairs (limited scale). The EKS-method then is applicable, whilst the GK-method is not. On the other hand meth-

c) Balanced method of van Yzeren
d) Other block methods

(8.3.29) $$P_{AB}^{ECLA} = \frac{\sum_{k=1}^{n} p_{kB}\left[\sum_{i=1}^{m} q_{ki}\right]}{\sum_{k=1}^{n} p_{kA}\left[\sum_{i=1}^{m} q_{ki}\right]} = \frac{\sum_{k=1}^{n} p_{kB} Q_k}{\sum_{k=1}^{n} p_{kA} Q_k} = \frac{\sum_{k=1}^{n} p_{kB} \bar{q}_k}{\sum_{k=1}^{n} p_{kA} \bar{q}_k} .$$

Thus prices in the numerator (p_{kB}) and in the denominator (p_{kA}) are weighted with the same quantities referring to the total block of m countries (in contrast to weights, q_{iA} or q_{iB} respectively, in the Laspeyres - or Paasche approach). Obviously weights $\bar{q}_k$ guarantee transitivity in the same way in which a *Lowe index* P^{LW} satisfies the circular test in intertemporal comparison of any three points

(8.3.30) $$P_{0t}^{LW} = P_{0s}^{LW} P_{st}^{LW} = \frac{\sum p_s \bar{q}}{\sum p_0 \bar{q}} \frac{\sum p_t \bar{q}}{\sum p_s \bar{q}} = \frac{\sum p_t \bar{q}}{\sum p_0 \bar{q}} ,$$

since weights in P^{LW} are common to all periods, not depending on either of these periods, 0, s or t.

8.3. Averaging methods for multinational comparisons

a) The EKS-method (formula and interpretation)	b) Caves-Christensen-Diewert (CCD) method

a) The EKS-method
Formula and interpretation

(8.4.1) $$P_{AB}^{EKS} = \left[\prod_{i}^{m} P_{iB}^{F} P_{Ai}^{F}\right]^{1/m} = \left[\prod_{i}^{m} \frac{P_{Ai}^{F}}{P_{Bi}^{F}}\right]^{1/m} = \left[\left(P_{AB}^{F}\right)^2 \prod_{i\neq A} P_{Ai}^{F} \prod_{i \neq B} P_{iB}^{F}\right]^{1/m} .$$

Equation 8.4.1 also shows that P_{AB} depends not only on prices and quantities of the countries i = A and i = B, but on all other m-2 countries. Thus if the set of countries is extended all price indices must be recalculated. The EKS method therefore is a closed system of parities.

How to read eq. 8.4.1? Two countries (m = 2), A and B only:

A product of *two* factors has to be calculated, the first factor accounting for i = A and the second for i = B to get $P_{AB}^{EKS} = \left[\left(P_{AB}^{F} P_{AA}^{F}\right)\left(P_{BB}^{F} P_{AB}^{F}\right)\right]^{1/2} = P_{AB}^{F}$.

Hence with only two countries involved the EKS-parity equals the Fisher-parity.

Three countries (m = 3), A, B and C (each parity a product of *three* factors):

$$P_{AB}^{EKS} = \left[\underbrace{P_{AB}^{F} P_{AA}^{F}}_{i=A} \underbrace{P_{BB}^{F} P_{AB}^{F}}_{i=B} \underbrace{P_{CB}^{F} P_{AC}^{F}}_{i=C}\right]^{1/3}$$ reduces to $$P_{AB}^{EKS} = \sqrt[3]{\left(P_{AB}^{F}\right)^2 P_{AC}^{F} P_{CB}^{F}}$$

using identity and time reversibility of Fisher-indices. Analogously the two remaining parities

$$P_{AC}^{EKS} = \left[\left(P_{AC}^{F} P_{AA}^{F}\right)\left(P_{BC}^{F} P_{AB}^{F}\right)\left(P_{CC}^{F} P_{AC}^{F}\right)\right]^{1/3} = \sqrt[3]{\left(P_{AC}^{F}\right)^2 P_{AB}^{F} P_{BC}^{F}}$$

$$P_{BC}^{EKS} = \left[\left(P_{AC}^{F} P_{BA}^{F}\right)\left(P_{BC}^{F} P_{BB}^{F}\right)\left(P_{CC}^{F} P_{BC}^{F}\right)\right]^{1/3} = \sqrt[3]{\left(P_{BC}^{F}\right)^2 P_{BA}^{F} P_{AC}^{F}} .$$

EKS-parities pass the **time reversal** test, because interchanging of A and B in eq. 8.4.1 leads to

(8.4.2) $$P_{BA}^{EKS} = \left[\prod_{i=1}^{m} P_{iA}^{F} P_{Bi}^{F}\right]^{1/m} = \left[\prod_{i=1}^{m} \frac{1}{P_{iB}^{F}} \frac{1}{P_{Ai}^{F}}\right]^{1/m} = \frac{1}{P_{AB}^{EKS}} ,$$

using the country reversibility of the Fisher formula ($P_{iA} = 1/P_{Ai}$). Also **transitivity** holds

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ods utilizing price-index material only (like EKS) involve a somewhat nebulous concept of quantity and there are difficulties in allowing for a different size of the countries to be compared as well as for different quantities of commodities.

(8.4.3) $P_{AC}^{EKS} = \left[\prod_i^m P_{iC}^F P_{Ai}^F\right]^{1/m} = \left[\prod_i^m P_{iB}^F P_{Ai}^F\right]^{1/m}\left[\prod_i^m P_{iC}^F P_{Bi}^F\right]^{1/m} = P_{AB}^{EKS} P_{BC}^{EKS}$,

The EKS parity can be interpreted as geometric mean of all indirect comparisons between A and B through all possible link countries i = 1, ...,m (including A and B as link countries).

$$P_{AB}^{EKS} = \left[\underbrace{P_{AA}^F P_{AB}^F}_{\text{link A}} \underbrace{P_{AB}^F P_{BB}^F}_{\text{link B}} \underbrace{P_{AC}^F P_{CB}^F}_{\text{link C}}\right]^{1/3} \quad \text{since } \prod P_{iB}^F \prod P_{iB}^F = 1$$

This can easily be verified using the equations for the three country case $P_{AC}^{EKS} = \sqrt[3]{\left(P_{AC}^F\right)^2 P_{AB}^F P_{BC}^F} = P_{AB}^{EKS} P_{BC}^{EKS} = \sqrt[3]{\left(P_{AB}^F\right)^2 P_{AC}^F P_{CB}^F}\sqrt[3]{\left(P_{BC}^F\right)^2 P_{BA}^F P_{AC}^F}$ The RHS of this eq. of course is $\sqrt[3]{\left(P_{AB}^F\right)^2 P_{AC}^F \frac{1}{P_{BC}^F}\left(P_{BC}^F\right)^2 \frac{1}{P_{AB}^F} P_{AC}^F}$ which is in fact P_{AC}^{EKS}.

EKS parities also pass the **factor reversal test** (see below).[77]

Derivation of P^{EKS}

1. by generalizing the Fisher formula

Consider a single parity between any two countries i and j $P_{ij} = P_j/P_i$ such that

$$P_{ij}^L \frac{P_i}{P_j} = \frac{P_j}{P_i} P_{ji}^L = \frac{P_j}{P_i}\cdot\frac{1}{P_{ij}^P} \quad \text{solving for } P_{ij} \text{ gives } \left(P_j/P_i\right)^2 = P_{ij}^L P_{ij}^P = \left(P_{ij}^F\right)^2$$

A "natural" generalization using normalized country weighs $f_i = g_i/\Sigma g_i$ ($\Sigma f_i = 1$) is

(8.4.5) $\prod_i \left[P_{ij}^L\left(P_i/P_j\right)\right]^{f_i} = \prod_i \left[P_{ji}^L\left(P_j/P_i\right)\right]^{f_i}$ (for i = 1, ...,m) leading to

(8.4.6) $P_j/P_i = \prod_k \left(P_{kj}^F P_{ik}^F\right)^{f_k} = P_{ij}^{GEKS}$

which is the generalized EKS-parity (or GEKS) in the unweighted case of $f_k = 1/m$ we get the "normal" EKS solution of eq. 8.4.1.

To see this consider first the situation with j fixed and i taking on all values 1, 2, ..., m, hence

$\left[P_{1j}^L\left(P_1/P_j\right)\right]^{f_1}\left[P_{2j}^L\left(P_2/P_j\right)\right]^{f_2}...\left[P_{mj}^L\left(P_m/P_j\right)\right]^{f_m} = \left[P_{j1}^L\left(P_j/P_1\right)\right]^{f_1}\left[P_{j2}^L\left(P_j/P_2\right)\right]^{f_2}...\left[P_{jm}^L\left(P_j/P_m\right)\right]^{f_m}$ yielding

(8.4.7) $\left(P_{1j}^F\right)^{f_1}\left(P_{2j}^F\right)^{f_2}...\left(P_{mj}^F\right)^{f_m} = P_j/P_1^{f_1}P_2^{f_2}...P_m^{f_m}$. Now consider eq. 5 if i is fixed

$\left[P_{i1}^L\left(P_i/P_1\right)\right]^{f_1}\left[P_{i2}^L\left(P_i/P_2\right)\right]^{f_2}...\left[P_{im}^L\left(P_i/P_m\right)\right]^{f_m} = \left[P_{1i}^L\left(P_1/P_i\right)\right]^{f_1}\left[P_{2i}^L\left(P_2/P_i\right)\right]^{f_2}...\left[P_{im}^L\left(P_m/P_i\right)\right]^{f_m}$ giving

(8.4.8) $\left(P_{i1}^F\right)^{f_1}\left(P_{i2}^F\right)^{f_2}...\left(P_{im}^F\right)^{f_m} = P_1^{f_1}P_2^{f_2}...P_m^{f_m}/P_i$, upon multiplication of eqs. 7 with 8 we get

(8.4.6a) $\left(P_{i1}^F P_{1j}^F\right)^{f_1}...\left(P_{im}^F P_{mj}^F\right)^{f_m} = \frac{P_j}{P_i} = P_{ij}^{GEKS}$.

EKS (and GEKS) parities pass the factor reversal test. Quantity indices can be obtained by interchanging prices and quantities in the price index formula (eq. 8.4.1a)

$$Q_{AB}^{EKS} = \sqrt[3]{\left(Q_{AB}^F\right)^2 Q_{AC}^F Q_{CB}^F} \rightarrow P_{AB}^{EKS} Q_{AB}^{EKS} = \sqrt[3]{\left(\frac{\Sigma p_B q_B}{\Sigma p_A q_A}\right)^2 \frac{\Sigma p_C q_C}{\Sigma p_A q_A}\frac{\Sigma p_B q_B}{\Sigma p_C q_C}} = \frac{\Sigma p_B q_B}{\Sigma p_A q_A}$$

---

77 This can no longer be assumed if P^F is replaced by another index function such as Törnquist P^T (in the CCD method, see below) or P^{ST} in Banerjee´s factorial approach functions.

2. by minimizing a distance

The general distance minimization criterion reads as follows[78]

(8.4.9) $$\min \Delta = \min \Delta(P_1,...,P_m) = \min_{P_1,...,P_m} \sum_i \sum_k g_i g_k \left[\ln\left(P_{ik}^F\right) - \ln\left(P_k/P_i\right)\right]^2$$

The function $\Delta = g_1 g_2 \left\{\left(\ln P_{12}^F\right)^2 - 2\ln P_{12}^F\left(\ln P_2 - \ln P_1\right) + \left(\ln P_2 - \ln P_1\right)^2\right\} +$
$g_1 g_3 \left\{\left(\ln P_{13}^F\right)^2 - 2\ln P_{13}^F\left(\ln P_3 - \ln P_1\right) + \left(\ln P_3 - \ln P_1\right)^2\right\} + \ldots$

has to be differentiated with respect to P_1, P_2, ..., and set equal to zero[79]. Thus $\frac{\partial \Delta}{\partial P_1} = 0$ is leading after division by $2g_1/P_1$ and using $\ln\left(P_{11}^F\right) = \ln(1) = 0$ to $g_2\left(\ln \frac{P_{12}^F}{P_{21}^F}\right) + g_3\left(\ln \frac{P_{13}^F}{P_{31}^F}\right) + \ldots +$
$2\ln P_1\left(g_2 + g_3 + \ldots\right) = 2\left(g_2 \ln P_2 + g_3 \ln P_3 + \ldots\right)$ or simply

(8.4.10) $$\ln P_1 = \sum_{i=1}^m g_i \ln P_i - \sum_{i=1}^m g_i \ln\left(P_{1i}^F\right).$$ In a similar vein examining $\frac{\partial \Delta}{\partial P_2} = 0$ leads to

(8.4.11) $$\ln P_2 = \sum_{i=1}^m g_i \ln P_i - \sum_{i=1}^m g_i \ln\left(P_{2i}^F\right).$$ Subtraction of 10 from 11 yields

(8.4.1a) $$P_{12}^{GEKS} = \frac{P_2}{P_1} = \frac{\prod_i^m \left(P_{i2}^F\right)^{g_i}}{\prod_i^m \left(P_{i1}^F\right)^{g_i}} = \prod_{i=1}^m \left(\frac{P_{i2}^F}{P_{i1}^F}\right)^{g_i}$$

b) The Caves-Christensen-Diewert (CCD) - method

As Fisher's index formula so is the Törnqvist index country reversible but not transitive, i.e. the product $P_{AB}^T P_{BC}^T = \prod_{k=1}^n \left(\frac{p_{kB}}{p_{kA}}\right)^{\overline{w}_{AB}} \prod_{k=1}^n \left(\frac{p_{kC}}{p_{kB}}\right)^{\overline{w}_{BC}}$ where $\overline{w}_{AB} = (w_{kA}+w_{kB})/2$, and $\overline{w}_{BC}$, $\overline{w}_{AC}$ correspondingly not necessarily equals $P_{AC}^T = \prod_{k=1}^n \left(\frac{p_{kC}}{p_{kA}}\right)^{\overline{w}_{AC}}$ unless $\overline{w}_{AB} = \overline{w}_{BC} = \overline{w}_{AC}$.

A simple method to guarantee transitivity is to take an average of any two-countries-comparison as done in the Caves-Christensen-Diewert (CCD) index, recommended for international price comparisons. In analogy to the EKS system of parities

(8.4.1) $$P_{AB}^{EKS} = \left(\frac{\prod P_{Ai}^F}{\prod P_{Bi}^F}\right)^{i/m} = \left[\prod_i^m \frac{P_{Ai}^F}{P_{Bi}^F}\right]^{1/m}$$ the CCD index is defined as

(8.4.12) $$P_{AB}^{CCD} = \left(\frac{\prod_{i=1}^m \left(P_{Ai}^T\right)}{\prod_{i=1}^m \left(P_{Bi}^T\right)}\right)^{1/m} = \left(\prod_{i=1}^m \frac{P_{Ai}^T}{P_{Bi}^T}\right)^{1/m}$$ or equivalently

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[78] The logarithmic distance functions Δ is introduced in order to make the resulting multilateral transitive indices deviate the least from the non-transitive binary indices.

[79] I saw no proof spelled out in detail in the relevant literature. So I demonstrated the details of the proof in my book "Index Theory and Price Statistics".

(8.4.12a) $\ln\left(P_{AB}^{CCD}\right) = \frac{1}{m}\sum_{i=1}^{m}\ln\left(P_{Ai}^{T}\right) - \frac{1}{m}\sum_{i=1}^{m}\ln\left(P_{Bi}^{T}\right)$. Obviously P^{CCD} in fact is transitive since

$$P_{AC}^{CCD} = \left(\prod_{i=1}^{m}\frac{P_{Ai}^{T}}{P_{Ci}^{T}}\right)^{1/m} = \left(\prod_{i=1}^{m}\frac{P_{Ai}^{T}}{P_{Bi}^{T}}\right)^{1/m}\left(\prod_{i=1}^{m}\frac{P_{Bi}^{T}}{P_{Ci}^{T}}\right)^{1/m} = P_{AB}^{CCD}P_{BC}^{CCD}.$$

To demonstrate this in the m = 3 country case examine the matrix **T** of logarithms of parities

$$\mathbf{T} = \begin{bmatrix} 0 & \ln\left(P_{AB}^{T}\right) & \ln\left(P_{AC}^{T}\right) \\ -\ln\left(P_{AB}^{T}\right) & 0 & \ln\left(P_{BC}^{T}\right) \\ -\ln\left(P_{AC}^{T}\right) & -\ln\left(P_{BC}^{T}\right) & 0 \end{bmatrix}$$ then eq. 12a applied to $\ln\left(P_{AC}^{CCD}\right)$ gives

$3\ln\left(P_{AC}^{CCD}\right) = \sum_{i=1}^{m}\ln\left(P_{Ai}^{T}\right) - \sum_{i=1}^{m}\ln\left(P_{Ci}^{T}\right) = \ln\left(P_{AB}^{T}\right) + \ln\left(P_{AC}^{T}\right) - \left(-\ln\left(P_{AC}^{T}\right) - \ln\left(P_{BC}^{T}\right)\right)$ or $\ln\left(P_{AC}^{CCD}\right) =$

$\frac{2}{3}\ln\left(P_{AC}^{T}\right) - \frac{1}{3}\left[-\ln\left(P_{BC}^{T}\right) - \ln\left(P_{AB}^{T}\right)\right]$ that is $P_{AC}^{CCD} = \sqrt[3]{\left(P_{AC}^{T}\right)^{2}P_{AB}^{T}P_{BC}^{T}}$ just like eq. 8.2.5a.

Literature

Two books of the author

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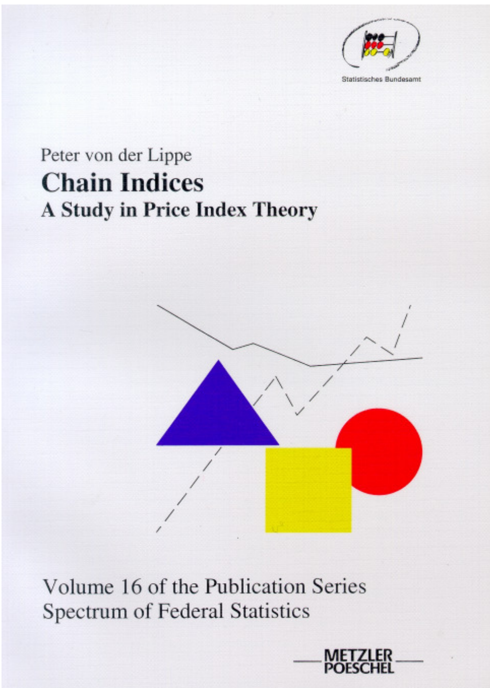
<td>
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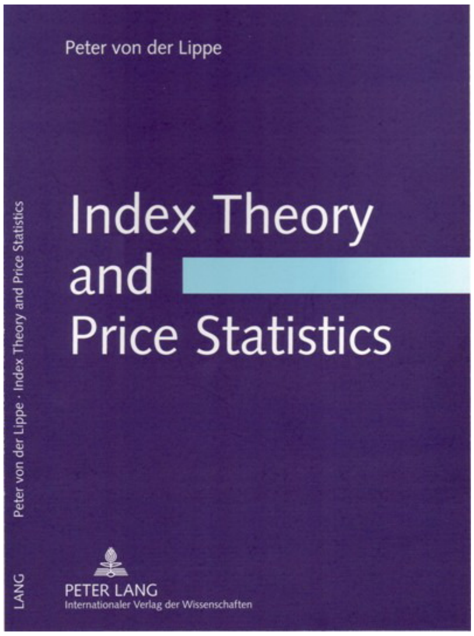
<td>
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