The Interpretation of Unit Value Indices

Price- and Unit-Value-Indices in Germany

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Agenda

1. **Setting the stage**
   motivation, definitions, terminology

2. **Drobisch Index (P^D) and other indices**
   (all-items unit value index) compared to the "normal" Paasche and Laspeyres index

3. **"Drobisch-Paasche" or "hybrid Paasche" index compared to the normal Paasche index** (shows that difference is resulting from structural changes, and can be explained in terms of covariances using a generalized theorem of L. v. Bortkiewicz)

4. **Drobisch-Paasche index and the normal Laspeyres index**
   (interpretation in terms of covariances and the L- and S-effect)

5. **Conclusion**

- **Literature** **(UVIs cannot replace price indices)**

  Balk 1994, 1995 (1998), 2005

  Diewert 1995 (NBER paper), **2004** etc., in particular **2010** (="Notes on Unit Value Bias", unpublished, Aug. 2010)

  Parniczky (1974)

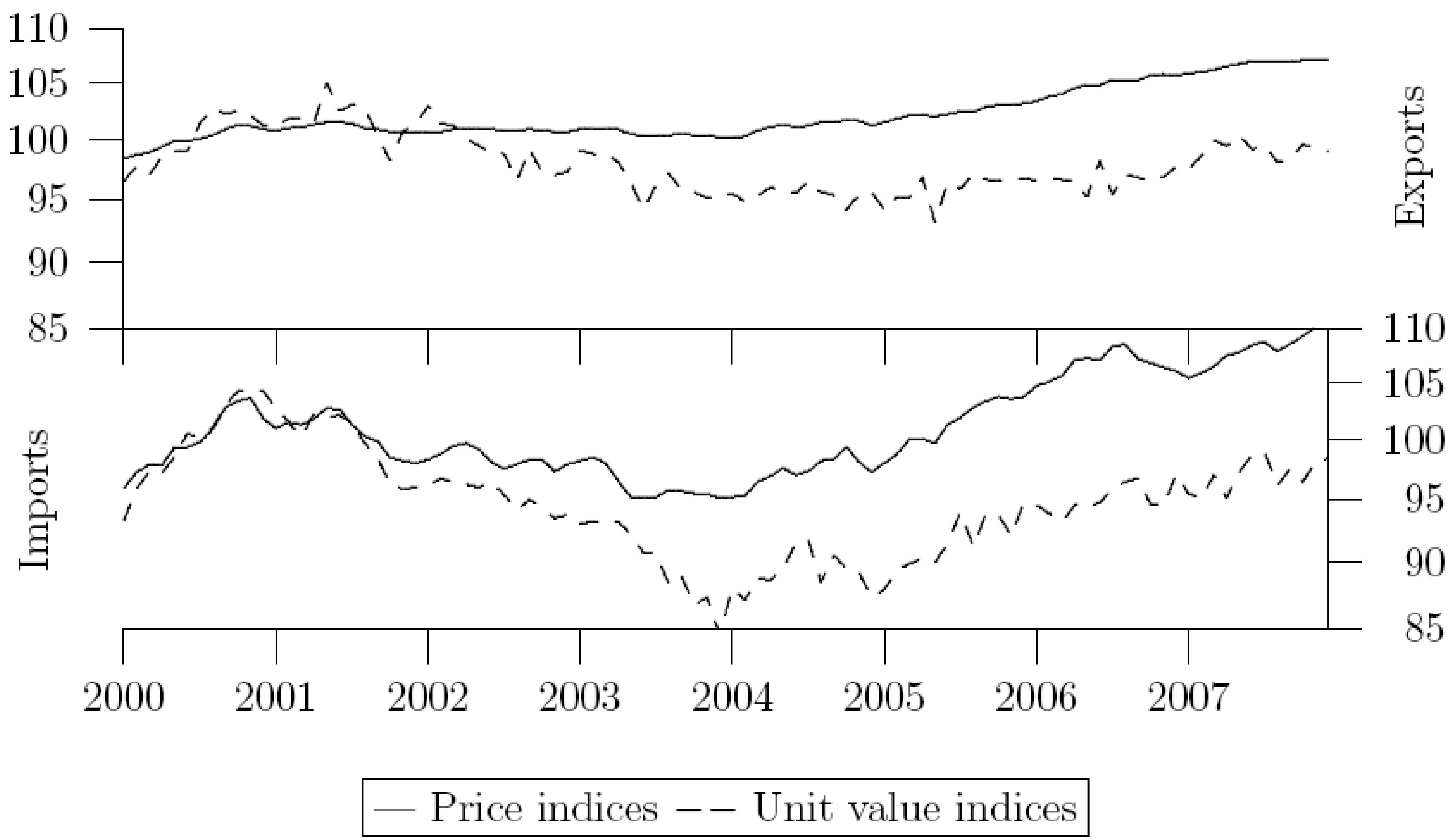
  Silver (2007) Do Unit Value Export, Import, and Terms of Trade Indices Represent or Misrepresent Price Indices, IMF Working Paper WP/07/121

  von der Lippe 2006 submitted to GER (also "Diskussionsbeiträge…") http://mpra.ub.uni-muenchen.de/5525/1/MPRA_paper_5525.pdf

  2010 Ottawa Group revision of a 2009 paper (for the 11th Meeting)

  http://mpra.ub.uni-muenchen.de/24743/1/MPRA_paper_24743.pdf

1. Setting the Stage 1.1. Introduction and Motivation



- **One-Stage and Two-Stage Index Compilation (TSC)**
  - $k = 1,2,\ldots,K$ CNs
  - $j = 1,2,\ldots,n_k$ commodity within a CN
  - prices p_{kjt} quantities q_{kjt} $t = 0, 1$

Aggregation in two stages;
$\Sigma n_k = n$
(all items)
use 3 subscripts

- **Unit values (Durchschnittswerte)**

all items

$$(1)\quad \tilde{p}_t = \frac{\sum_k \sum_j p_{kjt} q_{kjt}}{\sum_k \sum_j q_{kjt}} = \frac{\sum_k \sum_j p_{kjt} q_{kjt}}{Q_t} = \sum_k \sum_j p_{kjt} \frac{q_{kjt}}{Q_t} = \sum_k \sum_j p_{kjt} s_{kjt}$$

for the k-th CN

$$(2)\quad \tilde{p}_{kt} = \frac{\sum_j p_{kjt} q_{kjt}}{\sum_j q_{kjt}} = \sum_{j=1}^{n_k} p_{kjt} \frac{q_{kjt}}{Q_{kt}} = \sum p_{kjt} m_{kjt}$$

quantity share weights

$m_{kjt} \neq s_{kjt}$

later (also related to Q_t)

$\sigma_{kt} = Q_{kt}/Q_t$

$s_{kjt} = m_{kjt}\sigma_{kt}$

- **Covariance**

  - **all items**

(3) $$\mathrm{Cov}(x,y,w)=\sum\sum\left(x_{kjt}-\bar{x}\right)\left(y_{kjt}-\bar{y}\right)w_{kj}=$$

$$=\sum\sum x_{kjt}y_{kjt}w_{kj}-\bar{x}\cdot\bar{y} \qquad \sum_k\sum_j w_{kj}=1$$

known as "shift theorem"

  - **k-th CN**

(3a) $$\mathrm{cov}_k(x,y,w^*)=\sum_{j=1}^{n_k}\left(x_{kjt}-\bar{x}_k\right)\left(y_{kjt}-\bar{y}_k\right)w^*_{kj}$$

$$\sum_j w^*_{kj}=1$$

- **All-items-index of unit values**

(Drobisch [price] index)

$$(4)\quad P_{01}^{D} = \frac{\sum_k \sum_j p_{kj1} q_{kj1} / \sum_k \sum_j q_{kj1}}{\sum_k \sum_j p_{kj0} q_{kj0} / \sum_k \sum_j q_{kj0}} = \frac{Q_0}{Q_t} \frac{\sum_k \sum_j p_{kj1} q_{kj1}}{\sum_k \sum_j p_{kj0} q_{kj0}} = \frac{V_{01}}{Q_1 / Q_0} = \frac{\tilde{p}_1}{\tilde{p}_0}$$

This index P^D is widely known as "unit value index"
(better: Drobisch index)
In practice P^D cannot be compiled due to Q_0 and Q_1

however, Q_{kt} can be meaningfully established, thus also $\tilde{p}_{k1} / \tilde{p}_{k0}$

- **There is also a Drobisch quantity index** (not less problematic and likewise irrelevant in practice)

$$(4a)\quad Q_{01}^{D} = \tilde{Q}_{01} = Q_1 / Q_0 \qquad \text{note that} \qquad V_{01} = P_{01}^{D} \tilde{Q}_{01}$$

- There is another TSC-index *actually compiled* in official statistics (e.g. German foreign trade statistics)

(5) $$PU_{01}^{P}=\frac{\sum_k \tilde{p}_{k1}Q_{k1}}{\sum_k \tilde{p}_{k0}Q_{k1}}=\frac{\sum_k^K\sum_j^{n_k} p_{kj1}q_{kj1}}{\sum_k^K Q_{k1}\left(\sum_j^{m_k}\frac{p_{kj0}q_{kj0}}{Q_{k0}}\right)}=\frac{\sum_k^K\sum_j^{n_k} p_{kj1}q_{kj1}}{\sum_k^K Q_{k1}\left(\sum_j^{n_k} p_{kj0}m_{kj0}\right)}$$

This index is also known as "unit value index". It is a TSC-Paasche price index **using unit values** instead of prices **as building blocs** (on the first stage).
To avoid confusion with P^D how should it be called?

- Drobisch-Paasche
- hybrid Paasche (HP)
- Paasche (price) index of unit-values (PU^P)

other indices on the basis of unit values

$$PU_{01}^{L}=\sum\tilde{p}_{k1}Q_{k0}\Big/\sum\tilde{p}_{k0}Q_{k0}=\sum\tilde{p}_{k1}Q_{k0}\Big/\sum\sum p_{kj0}q_{kj0}$$

$$QU_{01}^{P}=\sum Q_{k1}\tilde{p}_{k1}\Big/\sum Q_{k0}\tilde{p}_{k1}$$

or QU^L, PU^F, QU^F etc.

1.4. Indices to be compared (+ next steps in the presentation)

- **all-items unit value index (= Drobisch index)**
  compared with Paasche, Laspeyres (+ Fisher) ⟶ **section 2**

- **PU^P index** (hybrid Paasche or **Paasche index of unit values**)
  compared with Paasche, Laspeyres ⟶ **section 3** (more relevant as regards official price statistics)

(one-stage-, or pure) Paasche index (6) / … Laspeyres index (6a) resp.

$$(6)\quad P_{0t}^{P} = \frac{\sum_{k=1}^{K}\sum_{j=1}^{n_k} p_{kj1} q_{kj1}}{\sum_{k=1}^{K}\sum_{j=1}^{n_k} p_{kj0} q_{kj1}} = \frac{\sum_k \tilde{p}_{k1} Q_{k1}}{\sum_k \sum_j p_{kj0} q_{kj1}} \qquad P_{0t}^{L} = \frac{\sum_{k=1}^{K}\sum_{j=1}^{n_k} p_{kjt} q_{k0}}{\sum_{k=1}^{K}\sum_{j=1}^{n_k} p_{kj0} q_{kj0}} = \frac{\sum_k \sum_j p_{kjt} q_{kj0}}{\sum_k \tilde{p}_{k0} Q_{k0}}$$

in all comparisons a covariance plays a major part

$$PU_{01}^{P} = \sum \tilde{p}_{k1} Q_{k1} \Big/ \text{den.} \qquad PU_{01}^{L} = \text{num.} \Big/ \sum \tilde{p}_{k0} Q_{k0}$$

P and PU indices have <u>num</u>erator or <u>deno</u>minator in common

- **P^D is <u>not</u> simply a weighted mean of unit-value-relatives** (as PU^P and PU^L) much less a mean of price relatives (by contrast to P^L and P^P which are weighted means of price-relatives)

(7) $$P^D_{01} = \sum_k \frac{\tilde{p}_{k1}}{\tilde{p}_{k0}} \left(\frac{\tilde{p}_{k0}\sigma_{k1}}{\sum_k \tilde{p}_{k0}\sigma_{k0}} \right)$$

(7a) $$\sigma_{kt} = Q_{kt} \Big/ \sum_k Q_{kt} = Q_{kt} / Q_t$$

sum of weights $\neq 1$

- however, P^P and P^L **are** means of sub-indices (8)

$$P^P_{0t} = \frac{\sum_k \sum_j p_{kj1} q_{kj1}}{\sum_k \sum_j p_{kj0} q_{kj1}} = \sum_k P^{P(k)}_{01} \frac{\sum_j p_{kj0} q_{k1}}{\sum_k \sum_j p_{kj0} q_{k1}}$$

$$P^{P(k)}_{01} = \frac{\sum_j p_{kj1} q_{kj1}}{\sum_j p_{kj1} q_{kj1}}$$

in a similar vein

(8a) $$P^L_{0t} = \sum_k P^{L(k)}_{01} \frac{\sum_j p_{kj0} q_{k0}}{\sum_k \sum_j p_{kj0} q_{k0}}$$

Results found for "all-item" or "low level" P^D indices (sec. 2) cannot simply be translated into two-stage PU^P/PU^L indices (sec. 3), and the PU^P is not simply a more disaggregated variant of the Drobisch index P^D.

2. Drobisch index P^D and Paasche 2.2 covariance expressions (1)

- **Three Drobisch-Paasche biases** (according to **Diewert** (2010))

1. base period prices and change of quantity structure

$$\frac{P_{01}^{D}}{P_{01}^{P}} - 1 = \frac{n}{\tilde{p}_0} \cdot \mathrm{Cov}\left(p_{kj0}, s_{kj1} - s_{kj0}, 1/n\right) \qquad (9)$$

"unweighted" (= equal weights 1/n)

the relevant covariance is

$$\mathrm{Cov}(1) = \sum_k \sum_j \left(p_{kj0} - \bar{p}_0\right)\left(\{s_{kj1} - s_{kj0}\} - 0\right)\frac{1}{n}$$

unweighted mean of $s_{kj1} - s_{kj0}$ is 0 and of p_{kj0} is $\bar{p}_0 = \sum \sum p_{kj0} / n$

- **conditions** for **vanishing bias**

C1 all base-period prices equal

C2 quantity shares s remain constant (then also $P^D = P^L = P^P = P^F$)

C3 zero-covariance

2. base period prices and growth rates of quantity shares

(10) $$\frac{P_{01}^D}{P_{01}^P} - 1 = \frac{Cov(p_{kj0}, G_{kj}, s_{kj0})}{\tilde{p}_0}$$

weights are base period quantity shares s_{kj0}

$$G_{kj} = s_{kj1}/s_{kj0} - 1 \rightarrow \sum\sum s_{kj0}\frac{s_{kj1}}{s_{kj0}} = 1 \rightarrow \sum\sum s_{kj0}G_{kj0} = 0 \qquad \sum\sum s_{kj0}p_{kj0} = \tilde{p}_0$$

the relevant covariance therefore is

$$Cov(2) = \sum_k \sum_j (p_{kj0} - \tilde{p}_0)(G_{kj} - 0) \cdot s_{kj0}$$

Condition C1 amounts here to $p_{kj0} = \bar{p}_0 = \tilde{p}_0 \ \forall k, j$ also C2 is the same

3. base period prices and change of quantities

← this is the formula of my "S-effect"

(11) $$\frac{P_{01}^D}{P_{01}^P} - 1 = \frac{Cov(p_{kj0}, q_{kj1}/q_{kj0}, s_{kj0})}{\tilde{p}_0 \tilde{Q}_{01}}$$

also Parniczky's formula

note (11a): $\sum\sum s_{kj0}\frac{q_{kj1}}{q_{kj0}} = \tilde{Q}_{01}$

the relevant covariance now is

$$Cov(3) = \sum_k \sum_j (p_{kj0} - \tilde{p}_0)(q_{kj1}/q_{kj0} - \tilde{Q}_{01}) \cdot s_{kj0}$$

Diewert's three covariance expressions are closely related. Using

$\frac{q_{kj1}}{q_{kj0}} = (G_{kj}+1)\tilde{Q}_{01}$ and the shift theorem we get

$$Cov(2) = \sum\sum p_{kj0} G_{kj} s_{kj0} - \tilde{p}_0 \cdot 0 = \sum\sum p_{kj0} G_{kj} s_{kj0}$$

$$Cov(3) = \sum\sum p_{kj0} \frac{q_{kj1}}{q_{kj0}} s_{kj0} - \tilde{p}_0 \tilde{Q}_{01}$$

$$= \tilde{Q}_{01} \sum\sum p_{kj0} G_{kj} s_{kj0} + \tilde{Q}_{01} \sum\sum p_{kj0} s_{kj0} - \tilde{Q}_{01} \tilde{p}_0$$

and since $\sum\sum s_{kj0} p_{kj0} = \tilde{p}_0$

we get $\frac{Cov(3)}{\tilde{Q}_{01}} = Cov(2)$ and therefore

$$\frac{P_{01}^D}{P_{01}^P} - 1 = \frac{Cov(p_{kj0}, G_{kj}, s_{kj0})}{\tilde{p}_0} = \frac{Cov(p_{kj0}, q_{kj1}/q_{kj0}, s_{kj0})}{\tilde{p}_0 \tilde{Q}_{01}}$$

basically the formulas tell the same story

eq.11 (Cov(.2.)) eq.12 (Cov(.3.))

- **Three Drobisch-Laspeyres biases** (according to **Diewert** (2010))

1. current period prices and change of quantity structure

counterpart to eq. 10 and Cov(.1.)

here also unweighted

$$(12)\quad \frac{P_{01}^{D}}{P_{01}^{L.}} - 1 = \frac{n \cdot \mathrm{Cov}\left(p_{kj1}, s_{kj1} - s_{kj0}, 1/n\right)}{\sum\sum p_{kj1} s_{kj0}}$$

note: a hybrid denominator, neither $\tilde{p}_0 = \sum\sum s_{kj0} p_{kj0}$ nor $\tilde{p}_1 = \sum\sum s_{kj1} p_{kj1}$

the relevant covariance now is

$$\mathrm{Cov}(1^*) = \sum_k \sum_j \left(p_{kj1} - \bar{p}_1\right)\left(s_{kj1} - s_{kj0}\right)\frac{1}{n}$$

note $\bar{p}_1 = \sum\sum p_{kj1}/n$

$\sum\sum\left(s_{kj1} - s_{kj0}\right)/n = 0$

- **conditions** for **vanishing bias**

C1* all **current**-period prices equal (C1: base period prices)

C2* = C2 quantity shares remain constant (then $P^D = P^L = P^P = P^F$)

C3 again: zero-covariance

2. current period prices and growth rates of *reciprocal* quantity shares

(13) $$\frac{P_{01}^L}{P_{01}^D} - 1 = \frac{Cov(p_{kj1}, \Gamma_{kj}, s_{kj1})}{\tilde{p}_1}$$

counter part to eq. 11 and Cov(.2.)

note: $P^L/P^D - 1$ whereas in (11) $P^D/P^P - 1$ inverse relation of (14) does not make sense

where $\Gamma_{kj} = s_{kj0}/s_{kj1} - 1$ and $\sum\sum \Gamma_{kj} s_{kj1} = 0$

Covariance Cov(.2*.)

$$Cov(2^*) = \sum_k \sum_j (p_{kj1} - \tilde{p}_1)(s_{kj0}/s_{kj1} - 1) \cdot s_{kj1} = \sum_k \sum_j (p_{kj1} - \tilde{p}_1)(\Gamma_{kj}) \cdot s_{kj1}$$

compare this covariance to

$$Cov(2) = \sum_k \sum_j (p_{kj0} - \tilde{p}_0)(G_{kj} - 0) \cdot s_{kj0}$$

where $G_{kj} = s_{kj1}/s_{kj0} - 1$ and $\sum\sum s_{kj0} G_{kj0} = 0$

3. current period prices and *reciprocal* change of quantities

$$(14)\quad \frac{P_{01}^{L}}{P_{01}^{D}}-1=\frac{Cov\left(p_{kj1},q_{kj0}/q_{kj1},s_{kj1}\right)}{\tilde{p}_1\left(\tilde{Q}_{01}\right)^{-1}}$$

counterpart to eq. 12 and Cov(.3.)

compare this covariance

$$Cov(3^*)=\sum_k\sum_j\left(p_{kj1}-\tilde{p}_1\right)\left(q_{kj0}/q_{kj1}-\left(\tilde{Q}_{01}\right)^{-1}\right)\cdot s_{kj1}$$

to

$$Cov(3)=\sum_k\sum_j\left(p_{kj0}-\tilde{p}_0\right)\left(q_{kj1}/q_{kj0}-\tilde{Q}_{01}\right)\cdot s_{kj0}$$

Note: we not only have reciprocal terms q_{kj0}/q_{kj1}, or Γ rather than G, we also study $P^L/P^D - 1$ (unlike P^D/P^P-1). After a digression: part 3: the practically more important study of indices for TSC (two-stage-compilations of index numbers)

Digression on axiomatics: The Drobisch index violates

- **commensurability**
- **proportionality (by implication: identity)**
- **mean value property** (cf. eq. 8 slide 10)

however P^D is able to pass the **time reversal test**

Another Digression

Symmetry in formulas for bias may be due to the time "antithetic" (Fisher) relation between Laspeyres and Paasche

$$(15)\quad V_{01} = \frac{\sum\sum p_1 q_1}{\sum\sum p_0 q_0} = \sum\sum\left(\frac{p_1}{p_0} - P_{01}^L\right)\left(\frac{q_1}{q_0} - Q_{01}^L\right)\cdot s_{kj0} + P_{01}^L Q_{01}^L$$

covariance in the theorem of L. v. Bortkiewicz

$$(15a)\quad \frac{1}{V_{01}} = \frac{\sum\sum p_0 q_0}{\sum\sum p_1 q_1} = \sum\sum\left(\frac{p_0}{p_1} - \frac{1}{P_{01}^P}\right)\left(\frac{q_0}{q_1} - \frac{1}{Q_{01}^P}\right)\cdot s_{kj1} + \frac{1}{P_{01}^P}\frac{1}{Q_{01}^P}$$

Paasche indices P_{01}

low level (first stage) goods	$p_{i1} = p_{kj1}$ (i=1,…,n) $p_{i0} = p_{kj0}$	↓
group of goods (CN)	↓	$\tilde{p}_{kt} = \frac{\sum_j p_{kjt} q_{kjt}}{\sum_j q_{kjt}}$ t = 0,1
second stage (weights)	q_{kj1}	$Q_{k1} = \Sigma q_{kj1}$
Index	$P_{0t}^P = \frac{\sum_k \tilde{p}_{k1} Q_{k1}}{\sum_k \sum_j p_{kj0} q_{kj1}}$	$PU_{01}^P = \frac{\sum_k \tilde{p}_{k1} Q_{k1}}{\sum_k \tilde{p}_{k0} Q_{k1}}$
name	(true) **Paasche**	**Drobisch-Paasche** ←

or
-Paasche index of unit values
- unit value index

the preferred name Paasche (price) index of unit values ⇒

$$\sum \tilde{p}_{k1} Q_{k1} = \sum \sum p_{kjt} q_{kjt}$$

3. Two-stage (hybrid) Paasche index PU^P 3.1 Introduction (1a)

Price indices based on

- **prices**
  - using price relatives or low-level price indices
  - $P_{01}^{L(k)}, P_{01}^{P(k)}$
  - → **price indices of Laspeyres/Paasche**
- **unit values**
  - all items unit values
    - $\tilde{p}_1 / \tilde{p}_0$
    - → **Drobisch index**
  - two stages (unit values on the first stage)
    - $\tilde{p}_{k1} / \tilde{p}_{k0}$
    - → **PU^P, PU^L index**

PU^P=Paasche (price) index of unit values

3. Drobisch-Paasche PU^P index   3.1 Introduction: some important facts (2)

1. PU^P is a weighted mean of unit-value-relatives, P^D is not

P^D is much less a mean of price relatives

$$PU_{01}^{P} = \sum_k \frac{\tilde{p}_{k1}}{\tilde{p}_{k0}} \frac{\tilde{p}_{k0} Q_{k1}}{\sum_k \tilde{p}_{k0} Q_{k1}} = \sum_k \frac{\tilde{p}_{k1}}{\tilde{p}_{k0}} \frac{\tilde{p}_{k0} \sigma_{k1}}{\sum_k \tilde{p}_{k0} \sigma_{k1}}$$

however (7) $$P_{01}^{D} = \sum_k \frac{\tilde{p}_{k1}}{\tilde{p}_{k0}} \left(\frac{\tilde{p}_{k0} \sigma_{k1}}{\sum_k \tilde{p}_{k0} \sigma_{k0}} \right)$$

PU^P is not simply a Drobisch index P^D on the basis of more homo-geneous sub-aggregates

2. PU^P is a mean of unit-value-relatives, while P^P is a mean of price relatives. Properties of unit value ratios as opposed price relatives (ratios of prices)

the ratio of unit values is not a mean of price relatives (16)

$$\frac{\tilde{p}_{k1}}{\tilde{p}_{k0}} = \frac{Q_{k0}}{Q_{k1}} \sum_j \frac{p_{kj1}}{p_{kj0}} \left(\frac{p_{kj0} q_{kj1}}{\sum_j p_{kj0} q_{kj0}} \right) = \sum_j \frac{p_{kj1}}{p_{kj0}} \left(\frac{p_{kj0} m_{kj1}}{\sum_j p_{kj0} m_{kj0}} \right)$$

unless the structure of quantities within each CN remains constant so that $m_{kj1} = m_{kj0}$

weights $p_{kj0} q_{kj1} / \tilde{p}_{k0} Q_{k1}$ add up to (16a) $Q_{01}^{L(k)} / \tilde{Q}_{01}^{k} = S_{01}^{k}$ ← for the S^k terms see eq. (20)

Furthermore ratios of unit values violate proportionality (hence also identity) and commensurability

weighted arithmetic mean of	yes	no
price relatives p_{kj1}/p_{kj0}	"normal" Paasche P^P (or Laspeyres P^L) **all price indices**	**ratios of unit values** (thus also PU^P and PU^L) unless $\sum_j \frac{p_{kj0}m_{kj1}}{\sum_j p_{kj0}m_{kj0}} = S_{01}^k = 1$ (that is no structural component) **Drobisch index** is not a mean of price relatives
ratios of unit values	PU^P and PU^L (**all indices of unit values**) but not "normal" price indices	P^P is not a mean of ratios of unit values $P_{01}^P = \sum_k \frac{\tilde{p}_{k1}}{\tilde{p}_{k0}} \frac{\tilde{p}_{k0}Q_{k1}}{\sum_k \sum_j p_{kj0}q_{kj1}}$ unless sum of weights is $QU^L/Q^L = 1/S = 1$ (again: if there is no structural component) **Drobisch index** is not a mean of ratios of unit values either

- Note: there are two PU indices, PU^P and PU^L, but only one Drobisch Index (one-stage or all-items unit value index) P^D.
- for practical reasons (German foreign trade statistic) in what follows we consider only PU^P (we don't compare PU^L to P^L)

PU^P compared to P^P

$$PU_{01}^{P} = \frac{\sum_k \tilde{p}_{k1} Q_{k1}}{\sum_k \tilde{p}_{k0} Q_{k1}} = \frac{\sum \tilde{p}_{k1} \sigma_{k1}}{\sum \tilde{p}_{k0} \sigma_{k1}}$$

$$(17) \quad \frac{PU_{01}^{P}}{P_{01}^{P}} - 1 = \frac{\sum_k \sum_j p_{kj0} q_{kj1}}{\sum_k \tilde{p}_{k0} Q_{k1}} - 1$$

$$= \frac{\sum_k Q_{k1} \sum_j p_{kj0} (m_{kj1} - m_{kj0})}{\sum_k \tilde{p}_{k0} Q_{k1}}$$

the numerator here is a covariance

PU^P compared to P^L

This comparison has more relevance, at least for Germany, because we have in this country customs based (census method) PU^P indices and survey based (sample) P^L indices.

However, a theory of the bias $\frac{PU_{01}^{P}}{P_{01}^{L}} - 1$ seems to be quite difficult (see **sec. 3.3**)

In (17) the term $\sum_j p_{kj0}(m_{kj1} - m_{kj0})$ is indeed a covariance

[cov_k type, within a CN, see (3a)]

(17a) $$\sum_j p_{kj0}(m_{kj1} - m_{kj0}) = n_k \, cov_k\left(p_{kj0}, m_{kj1} - m_{kj0}, 1/n_k\right)$$

$$cov_k(\ldots) = \sum_{j=1}^{n_k}\left(p_{kj0} - \bar{p}_{k0}\right)\left(m_{kj1} - m_{kj0}\right)\frac{1}{n_k} \quad \text{since} \quad \begin{matrix} \sum_j p_{kj10} = n_k \bar{p}_{k0} \\ \sum_j m_{kj1} = \sum_j m_{kj0} = 1 \end{matrix}$$

However, the bias $\frac{PU_{01}^P}{P_{01}^P} - 1$ is not a weighted average of these covariances

(17b) $$\frac{PU_{01}^P}{P_{01}^P} - 1 = \sum_k \frac{Q_{k1}}{\sum_k \tilde{p}_{k0} Q_{k1}} \cdot n_k \, cov_k\left(p_{kj0}, m_{kj1} - m_{kj0}, 1/n_k\right)$$

What matters is again covariance between **prices in 0** and **change of quantity structure** (now within the k^{th} CN

Diewert considered $P_{01}^P / PU_{01}^P - 1$ instead of $PU_{01}^P / P_{01}^P - 1$ and he found →

"In this section, we will find it convenient to define the bias using a reciprocal measure" (p. 13)

(18) $$P_{01}^{P} / PU_{01}^{P} - 1 = \frac{n \cdot Cov(p^0, s^0 - s^1, 1/n)}{\sum\sum p_{kj0} s_{kj0}}$$

reciprocal

p^0, s^0 und s^1 are vectors – of p_{kj0}, s_{kj0}, s_{kj1} - stacked up into a single n dimensional vector (using $m_{kjt}\sigma_{kj} = s_{kjt}$)

again: what matters is prices in 0, quantity change…

Diewert compared bias PU^P and P^D relative to P^P and PU^L and P^D relative to P^L. Our focus here, however, only PU^P relative to P^P and P^L

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v. d. Lippe's approach (2 points)

1. express discrepancy (= bias +1) as a weighted average of ratios of linear indices of CNs (sub-aggregates)

$$S = PU_{01}^{P} / P_{01}^{P} = \text{bias} + 1$$ S = structural component, "S-effect"

using the identity (19) $V_{01} = PU_{01}^{L} QU_{01}^{P} = PU_{01}^{P} QU_{01}^{L} = P_{01}^{L} Q_{01}^{P} = P_{01}^{P} Q_{01}^{L}$

we get (20) $$S = \frac{Q_{01}^{L}}{QU_{01}^{L}} = \sum_k \frac{Q_{01}^{L(k)}}{\tilde{Q}_{01}^{k}} \cdot \frac{\tilde{Q}_{01}^{k} s_{k0}}{\sum_k \tilde{Q}_{01}^{k} s_{k0}} = \sum_k S_{01}^{k} \cdot \frac{\tilde{p}_{k0} Q_{k1}}{\sum_k \tilde{p}_{k0} Q_{k1}}$$

notice $S = \sum_k \frac{Q_{01}^{L(k)}}{\tilde{Q}_{01}^k} \cdot \frac{\tilde{Q}_{01}^k s_{k0}}{\sum_k \tilde{Q}_{01}^k s_{k0}} = \sum_k S_{01}^k \cdot \frac{\tilde{p}_{k0} Q_{k1}}{\sum_k \tilde{p}_{k0} Q_{k1}}$ is a weighted mean of $S_{01}^k = \frac{Q_{01}^{L(k)}}{\tilde{Q}_{01}^k}$ terms which may be viewed as contributions of the k-th CN to the S-effect; and weights $\tilde{p}_{k0} Q_{k1} / \sum_k \tilde{p}_{k0} Q_{k1}$

2. As the S^k terms are ratios of **linear indices** you can make use of a **theorem of L. v. Bortkiewicz** (on the relation between two linear indices), which goes as follows ⇒ next slide

⇒ v.d.Lippe (2007), p. 194 for the Generalized Theorem, the famous special case is $X_t = P^P$ and $X_0 = P^L$

Ladislaus von Bortkiewicz (1923)

Theorem of L. v. Bortkiewicz

The ratio of two linear indices, X_t and X_0 respectively where (t = 1)

$$X_t = \frac{\sum x_t y_t}{\sum x_0 y_t} \quad \text{and} \quad X_0 = \frac{\sum x_t y_0}{\sum x_0 y_0} \quad \text{is given by} \quad \frac{X_t}{X_0} = 1 + \frac{s_{xy}}{\overline{X} \cdot \overline{Y}}$$

with the co-variance s_{xy}

$$s_{xy} = \sum \left(\frac{x_t}{x_0} - \overline{X} \right) \left(\frac{y_t}{y_0} - \overline{Y} \right) w_0 = \frac{\sum x_t y_t}{\sum x_0 y_0} - \overline{X} \cdot \overline{Y}$$

weights $w_0 = x_0 y_0 \big/ \sum x_0 y_0$

arithmetic means

$$\sum (x_t / x_0) \cdot w_0 = \overline{X} = X_0 \qquad \sum (y_t / y_0) \cdot w_0 = \overline{Y} = \sum y_t x_0 \big/ \sum y_0 x_0$$

Two covariance expressions to explain $S^{(k)}$ in eq. 18

$$(20)\quad S=\frac{Q_{01}^{L}}{QU_{01}^{L}}=\sum_k \frac{Q_{01}^{L(k)}}{\tilde{Q}_{01}^{k}}\cdot\frac{\tilde{p}_{k0}Q_{k1}}{\sum_k \tilde{p}_{k0}Q_{k1}}=\sum_k S_{01}^{k}\cdot\frac{\tilde{p}_{k0}Q_{k1}}{\sum_k \tilde{p}_{k0}Q_{k1}}$$

$$(21)\quad X_t = Q_{01}^{L(k)},\quad X_0=\tilde{Q}_{01}^{k}\qquad \sum\left(\frac{q_{kj1}}{q_{kj0}}-\tilde{Q}_{01}^{k}\right)\left(p_{kj0}-\tilde{p}_{k0}\right)\frac{q_{kj0}}{\sum q_{kj0}}$$

quantity shares m_{kj0} as weights

With this covariance **$c_k = cov_k(q_{kj1}/q_{kj0}, p_{kj0}, m_{kj0})$** - which bears some resemblance to the covariance $Cov(3) = Cov(q_{kj1}/q_{kj0}, p_{kj0}, s_{kj0})$ in eq. 11 - we get $c_k=\tilde{p}_{k0}\left(Q_{01}^{L(k)}-\tilde{Q}_{01}^{k}\right)$ and using the Bortkiewicz theorem

$$(21a)\quad S_{01}^{k}=\frac{Q_{01}^{L(k)}}{\tilde{Q}_{01}^{k}}=\frac{X_1}{X_0}=1+\frac{c_k}{\overline{X}\cdot\overline{Y}}=1+\frac{c_k}{\tilde{p}_{k0}\tilde{Q}_{01}^{k}}$$

and using (20)

3. Two-stage Paasche index PU^P 3.2 PU^P and Paasche P^P (6)

$$S = \frac{PU_{01}^P}{P_{01}^P} = \sum_k S_{01}^k \cdot \frac{\tilde{p}_{k0} Q_{k1}}{\sum_k \tilde{p}_{k0} Q_{k1}} = 1 + \frac{\sum_k c_k Q_{k0}}{\sum_k \tilde{p}_{k0} Q_{k1}}$$

or in terms of a bias

(21b)
$$\frac{PU_{01}^P}{P_{01}^P} - 1 = \sum_k \frac{Q_{k0}}{\sum_k \tilde{p}_{k0} Q_{k1}} \cdot \text{cov}_k \left(p_{kj0}, q_{kj1} / q_{kj0}, m_{kj0} \right)$$

which may be compared to the formula (17a) on slide 21

(17a)
$$\frac{PU_{01}^P}{P_{01}^P} - 1 = \sum_k \frac{Q_{k1}}{\sum_k \tilde{p}_{k0} Q_{k1}} \cdot \text{cov}_k \left(p_{kj0}, m_{kj1} - m_{kj0}, 1/n_k \right)$$

Using the shift theorem (3) it can be seen that in both equations the numerator amounts to $\sum_j p_{kj0} q_{kj1} + Q_{k1} \tilde{p}_{k0}$

Alternatively we might explain $(S^{(k)})^{-1}$

(22)

$X_t = \tilde{Q}_{01}^k$	$$\sum \left(\frac{q_{kj1}}{q_{kj0}} - Q_{01}^{L(k)} \right) \left(\frac{1}{p_{kj0}} - \frac{1}{\tilde{p}_{k0}} \right) \frac{p_{kj0} q_{kj0}}{\sum p_{kj0} q_{kj0}}$$
$X_0 = Q_{01}^{L(k)}$	

expenditure shares as weights

3. Digression PU^P PU^L and Drobisch P^D as ratios of two linear indices

PU^P relative to P^D

$$\frac{P_{01}^D}{PU_{01}^P} = \frac{QU_{01}^L}{\tilde{Q}_{01}} = \frac{X_1}{X_0} \qquad X_0 = \frac{\sum_k Q_{k1} \cdot 1}{\sum_k Q_{k0} \cdot 1} = \tilde{Q}_{01}$$

(23) $\mathrm{Cov}(Q_{k1}/Q_{k0}, \tilde{p}_{k0}, Q_{k0}/\sum Q_{k0})$ equivalently $\sum_k \left(Q_{k1}/Q_{k0} - \tilde{Q}_{01}\right)\left(\tilde{p}_{k0} - \tilde{p}_0\right)\frac{Q_{k0}}{Q_0}$

If above average **base period** unit values are associated with above average quantity changes P^D will be greater than PU^P

PU^L relative to P^D

$$\frac{P_{01}^D}{PU_{01}^L} = \frac{QU_{01}^P}{\tilde{Q}_{01}} = \frac{X_1}{X_0} \qquad X_1 = \frac{\sum_k Q_{k1}\tilde{p}_{k1}}{\sum_k Q_{k1}\tilde{p}_{k1}} = QU_{01}^P$$

what applied to the K **within** (the k CNs) covariances in eq. 21, now applies to the one **between** covariance

(23a) $\mathrm{Cov}(Q_{k1}/Q_{k0}, \tilde{p}_{k1}, Q_{k0}/\sum Q_{k0} = \sigma_{k0})$

If above average **current period** unit values …

PU^P and Laspeyres

(24) $$\frac{PU_{01}^P}{P_{01}^L}-1=\frac{\sum_k \tilde{p}_{k1}Q_{k1}}{\sum_k \tilde{p}_{k0}Q_{k1}}\cdot\frac{\sum_k \tilde{p}_{k0}Q_{k0}}{\sum_k\sum_j p_{kj1}q_{kj0}}-1$$

It appears much more difficult to compare these indices

compared to

(17) $$\frac{PU_{01}^P}{P_{01}^P}-1=\frac{\sum_k\sum_j p_{kj0}q_{kj1}}{\sum_k \tilde{p}_{k0}Q_{k1}}-1$$

than PU^P to P^P

or PU^L to P^L

we will therefore make a comparison in two steps (see slide

of little (or no) relevance $$\frac{PU_{01}^L}{P_{01}^L}-1=\frac{\sum_k \tilde{p}_{k1}Q_{k0}}{\sum_k\sum_j p_{kj1}q_{kj0}}-1$$

the equations are the counterparts to (20) and (22)

(20a) $$S^*=\frac{PU_{01}^L}{P_{01}^L}=\frac{Q_{01}^P}{QU_{01}^P}=\sum_k \frac{Q_{01}^{P(k)}}{\tilde{Q}_{01}^k}\cdot\frac{\sum_j p_{kj1}q_{ki0}}{\sum_k\sum_j p_{kj1}q_{ki0}}=\sum_k S_{01}^{*k}\cdot\frac{\sum_j p_{kj1}q_{ki0}}{\sum_k\sum_j p_{kj1}q_{ki0}}$$

relevant covariance to explain S*

(22a) $$\sum\left(q_{kj1}/q_{kj0}-\tilde{Q}_{01}^k\right)\left(p_{kj1}-\tilde{p}_{k1}^*\right)\cdot m_{kj0}$$

$$\tilde{p}_{k1}^*=\sum p_{kj1}m_{kj0}$$

also somehow "hybrid"

only 2nd factor (p -…) different

3. PU^P index 3.4 PU^P and P^P (summary of "structural effect")

1. Difference (bias) PU^P relative to P^P results from **structural** changes (structure of quantities), measured by (18) $S = PU^P/P^P = Q^L/QU^L$

2. S is a weighted mean of S_k measures ($Q^L/\tilde{Q}$ ratio of the k^{th} CN)*)

weights $\tilde{p}_{k0}Q_{k1} / \sum_k \tilde{p}_{k0}Q_{k1}$

*) numerator and denominator are **linear** indices therefore the theorem of L. v. Bortkiewicz applies

3. this theorem says that a **covariance** $cov_k(\ldots)$ is responsible for the contribution of the k^{th} CN to the S-effect (to S), and this covariance is

$$\sum \left(\frac{q_{kj1}}{q_{kj0}} - \tilde{Q}^k_{01} \right) \left(p_{kj0} - \tilde{p}_{k0} \right) \cdot m_{kj0}$$

and this means: **no S-effect** when

1. all prices in 0 equal $p_{kj0} = \tilde{p}_{k0} = \bar{p}_{k0}$
2. quantities remain constant $q_{kj1} = q_{kj0}$
3. covariance vanishes: above (below) average base period prices are associated with below (above) average increase in quantities
4. each $n_k = 1$ (homogenous CNs) more→

cf. conditions on slide 11 (regarding P^D and P^P) →

prices in 1 and p_{kj1}/p_{kj0} irrelevant →

Homogeneity of CNs

S-effect also vanishes if

1. no CNs, only individual goods
(or: each $n_k = 1$, perfectly homogeneous CNs)

2. all q_{kj1}/q_{kj0} equal (or = 1) **3.** all prices p_{kj0} equal $\forall j, k$
4. zero covariances or average of Cov_k weights $\tilde{p}_{k0}Q_{k1}/\sum_k \tilde{p}_{k0}Q_{k1}$

Economic interpretation of the S-effect

note: **prices in t are irrelevant** for the S-effect to occur by contrast to the L-effect (= substitution effect).

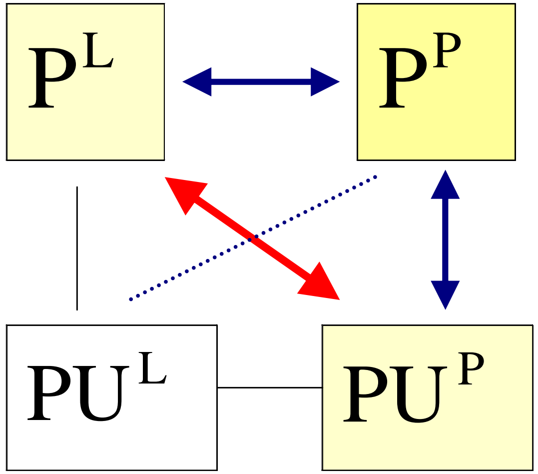
How to explain a quantity change although no price has changed?

Some consequences

condition "for choosing how to construct the subaggregates: in order to minimize bias (relative to the Paasche price index), use unit value aggregation over products that sell for the same price in the base period" (Diewert 2010, p. 14)

4. PU^P and P^L index (comparison in two steps, S- and L-effect)

Basis of decomposition $V_{0t} = PU^L_{0t} QU^P_{0t} = PU^P_{0t} QU^L_{0t}$



$$D = \frac{PU^P_{0t}}{P^L_{0t}} = \left(\frac{C}{Q^L_{0t} P^L_{0t}} + 1\right)\left(\frac{Q^L_{0t}}{QU^L_{0t}}\right) = \frac{P^P_{0t}}{P^L_{0t}} \cdot \frac{PU^P_{0t}}{P^P_{0t}} = L \cdot S$$

$$L = \frac{P^P_{0t}}{P^L_{0t}} = \frac{Q^P_{0t}}{Q^L_{0t}}$$

The covariance C here is (Theorem of Bortkiewicz)

$$C = \sum_i \left(\frac{p_{it}}{p_{i0}} - P^L_{0t}\right)\left(\frac{q_{it}}{q_{i0}} - Q^L_{0t}\right) \frac{p_{i0} q_{i0}}{\sum p_{i0} q_{i0}}$$

expenditure shares as weights rather than quantity shares

No L-effect (L = 1) if

1. all price relatives equal or = 1
2. all quantity relatives equal or =1
3. covariance = 0

both **quantity** change **and price** changes do not matter (price change irrelevant for the S-effect)

4. Why compare PUP to P^L rather than P^P Indices in Germany

Data source, conceptual differences

	Price index	Unit value index
Data	**Survey based** (monthly), **sample**; more demanding (weights!)	**Customs based** (by-product), **census**, Intrastat: survey
Formula	**Laspeyres**	**Paasche**
Quality ad-justment	**Yes**	**No** (feasible?)
Prices, aggregates	Prices of specific goods at time of **contracting**	Average value of CNs; time of **cross-ing border** CN = commodity numbers
New / dis-appearing goods	Included only when a new base period is defined; vanishing goods replaced by *similar* ones **constant selection of goods** *	Immediately included; price quotation of disappearing goods is simply discontinued **variable universe of goods**
Merits	Reflect **pure price** movement (ideally the same products over time)	"**Representativity**" inclusion of *all* products; **data** readily **available**
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* All price determining characteristics kept constant

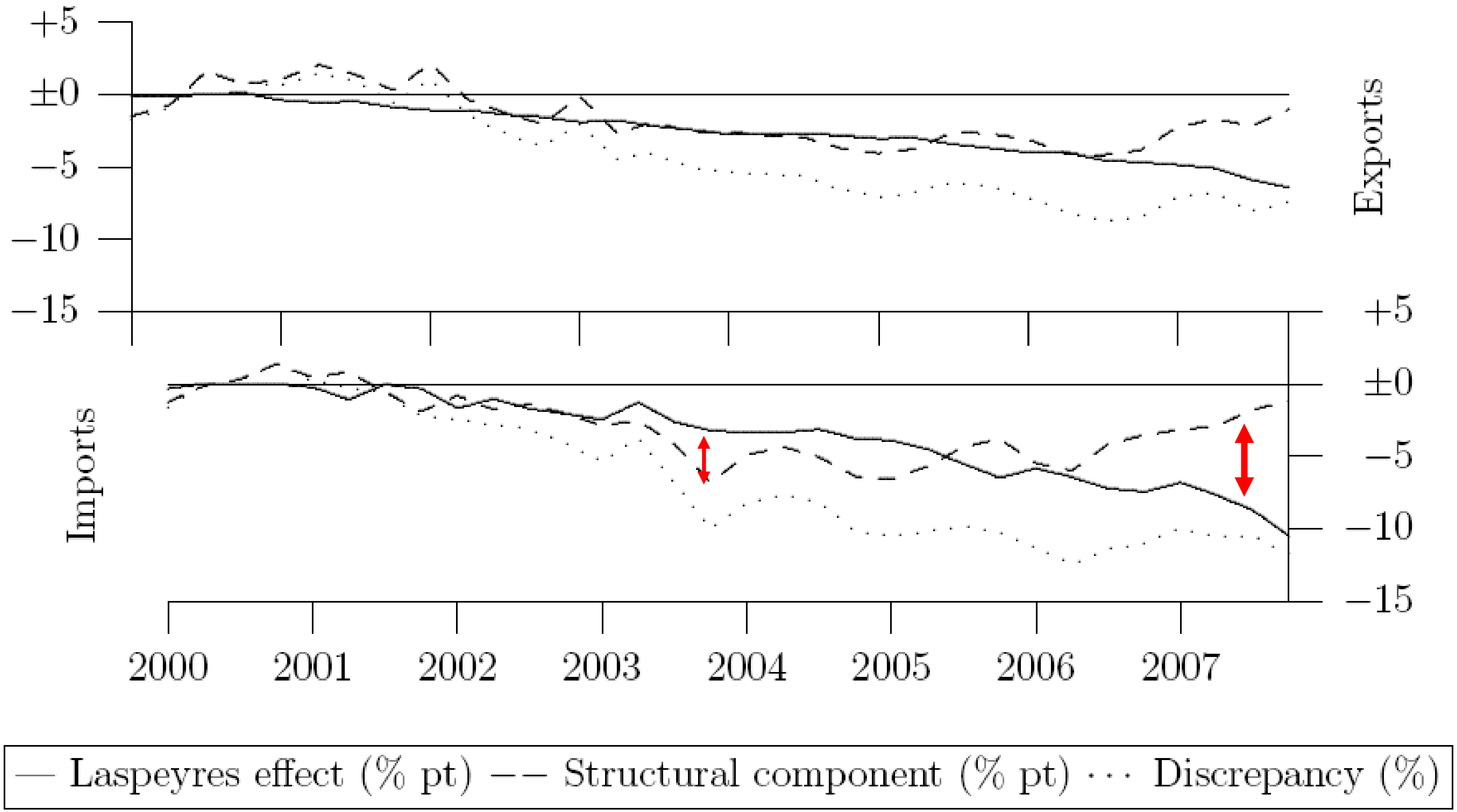
4. Hypothesis concerning comparison PU^P and P^L Empirical results for Germany

Hypothesis	Argument
1. U < P, growing **discrepancy**	Laspeyres (P) > Paasche (U) Formula of L. v. Bortkiewicz
2. Volatility U > P	U no pure price comparison (U reflecting changes in product mix [structural changes])
3. Seasonality U > P	U no adjustment for seasonally non-availability
4. U suffers from **heterogeneity**	Variable vs. constant selection of goods, CN less homogeneous than specific goods
5. Lead of P	Prices refer to the earlier moment of contracting (contract-delivery lag; exchange rates)
6. Smoothing (due to quality adjustment)	Quality adjustment in P results in smoother series

4. The two effects L and S

Deflator X and M respectively taken for P^P

are S and L independent components??



Data problems with updating of this figure

5. Conclusion: Problems and confusions with unit-value-indices

- Unit values as proxies for prices are increasingly important
- Unfortunately the term "unit value index" is used for very different index formulas
- The focus of index theory is almost exclusively on the practically less relevant index of Drobisch P^D (irrelevant because as a rule $\sum_k Q_{kt}$ does not exist)
- There is no consensus about the name of PU^P, PU^L
  (we should, however, find a name in order to stop the prevailing confusion)
- By contrast to P^D these indices are in fact weighted means of ratios of unit values (not of price relatives), and they make use of quantities Q_{kt} only
- The bias of PU^P relative to P^P can be explained by the covariance between base-period prices and changes in the structure of quantities.
- Many (interrelated) covariance-expressions are possible and the formulas are also a bit similar to formulas for the bias of P^D relative to P^P.