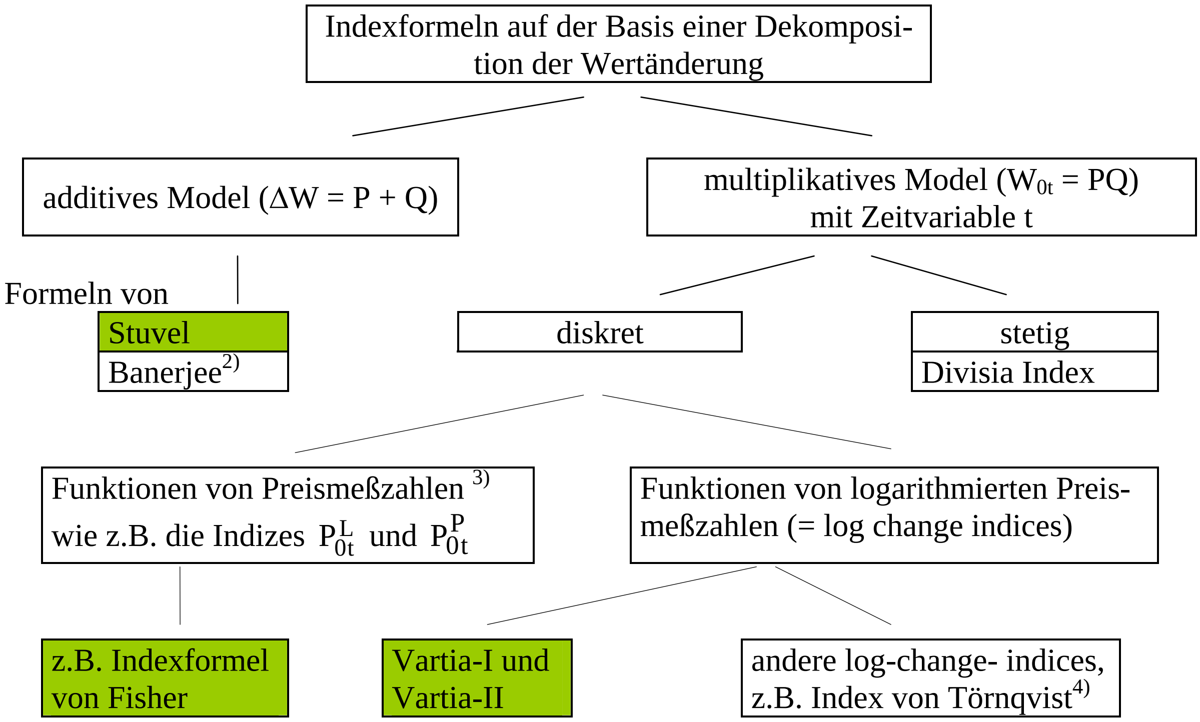
Log- change and ideal indices

Neuere Indexformeln, speziell "ideale" Indizes[1)]



1) "ideale" Indizes (durch Schattierung hervorgehoben) sind Indizes, die die Faktorumkehrbarkeit erfüllen.
2) auf Banerjee's factorial approach soll hier nicht weiter eingegangen werden.
3) als arithmetishe oder andere Mittelwerte (die meisten log-change indices kann man als geormetrische Mittel von Preismeßzahlen darstelllen).
4) auch Indizes, die nicht aus einem Dekompositionsmodell hergeleitet sind, wie z.B. der Cobb-Douglas index oder Verfeinerungen der Törnqvist Formel um besser Faktorumkehrbarkeitzu approximieren (Formeln von Theil, Sato usw.).

Formeln von Stuvel[1] (Preisindex P^{ST} und Mengenindex Q^{ST})

(1) $$P_{0t}^{ST} = \frac{P_{0t}^{L} - Q_{0t}^{L}}{2} + \sqrt{\left(\frac{P_{0t}^{L} - Q_{0t}^{L}}{2}\right)^2 + W_{0t}}$$

(2) $$Q_{0t}^{ST} = \frac{Q_{0t}^{L} - P_{0t}^{L}}{2} + \sqrt{\left(\frac{Q_{0t}^{L} - P_{0t}^{L}}{2}\right)^2 + W_{0t}}$$

---

[1] Sie erscheinen auch als Spezialfälle im Ansatz von Banerjee.

Growth rates and growth factors, despite being very popular, have the following two disadvantages:

- they are not symmetric, that is: $\frac{y_t - y_{t-1}}{y_{t-1}} \neq -\frac{y_{t-1} - y_t}{y_t}$ and
- the sum of two (or more) growth rates has no meaningful interpretation.

Furthermore, a general notion of growth rate could be as follows:

(3) $$\text{growthrate} = \frac{\text{absolute change } (\Delta y)}{\text{level } (A(y))}.$$

(4) Log-change (definition) $$D\ell_t = \ln\left(\frac{y_t}{y_{t-1}}\right) = \ln(y_t) - \ln(y_{t-1}) = \ln(f_t)$$

(5) Log-mean (definition) $$L(y_t, y_{t-1}) = L(y_{t-1}, y_t) = \frac{y_t - y_{t-1}}{\ln(y_t / y_{t-1})} \quad \text{if } y_t \neq y_{t-1}.$$

(6) $$D\ell_t = \ln\left(\frac{y_t}{y_{t-1}}\right) = r_t^L = \frac{y_t - y_{t-1}}{L(y_t, y_{t-1})},$$

Figure *1: Advantages of log changes over traditional growth rates

aspect	log changes	traditional growth rates[1)]
symmetry	$\ln(y_t) - \ln(y_{t-1}) = - [\ln(y_{t-1}) - \ln(y_t)]$	no symmetry
summation over successive intervals	$D\ell_t + D\ell_{t+1} = \ln\left(\frac{y_{t+1}}{y_{t-1}}\right)$ a growth related to a time span of *two* periods[2)]	the sum $r_t + r_{t+1}$ is not meaningful
eq. 6 interpretation	$D\ell_t = \frac{\Delta y}{A(y)} = \frac{y_t - y_{t-1}}{L(y_t, y_{t-1})}$ see eq. 6.4.5 log mean of y_t and y_{t-1} as "level"	lower (or upper) bound, that is y_{t-1} as level A(y) in the denominator

1) $r_t = (y_t - y_{t-1})/y_{t-1}$

2) correspondingly the sum of m adjacent $D\ell$ -terms measures a change over m periods

---

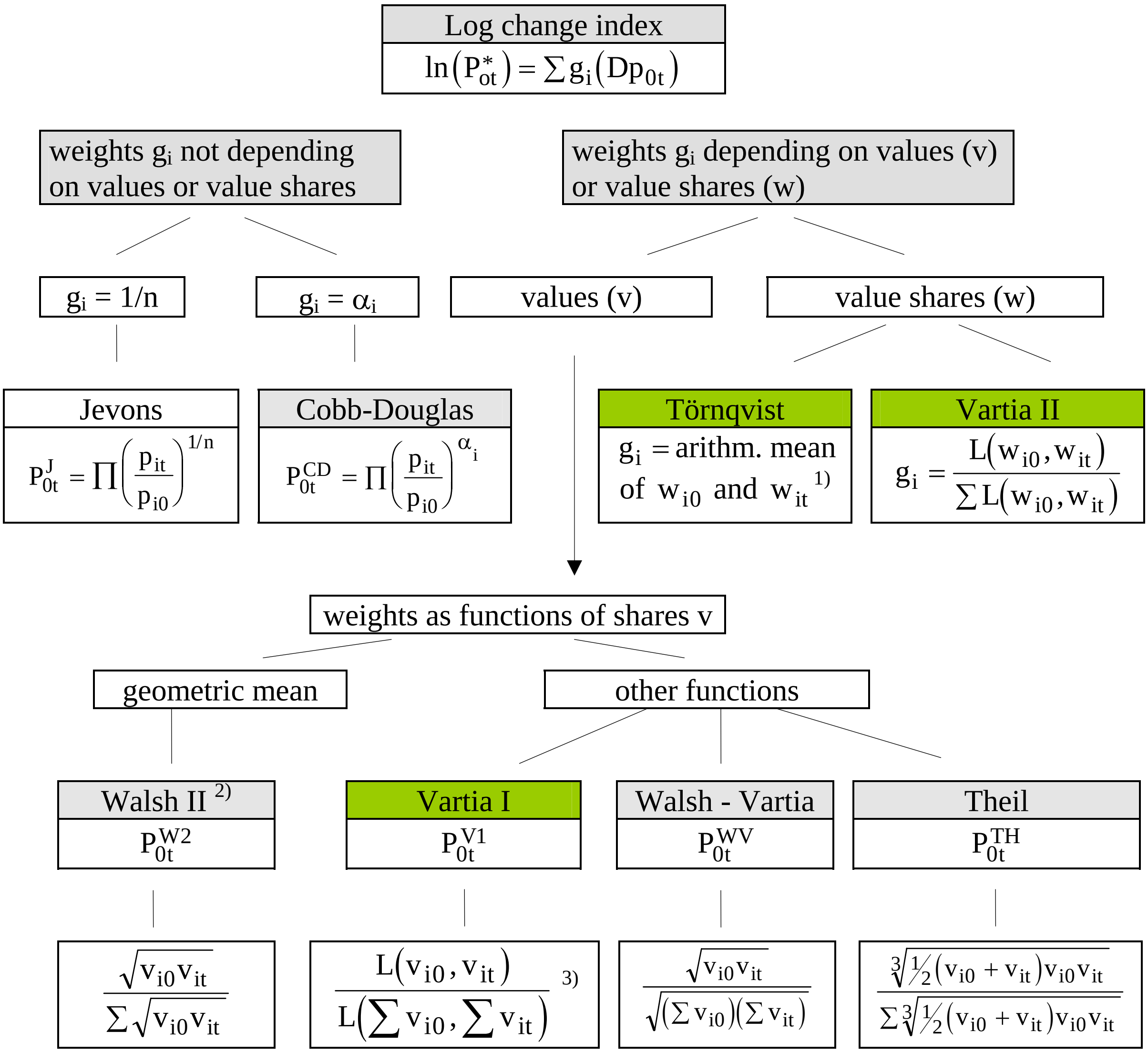
(7) $$Dp_i = \ln\left(\frac{p_{it}}{p_{i0}}\right) = \ln\left(\frac{p_{i1}}{p_{i0}}\right) + \ldots + \ln\left(\frac{p_{it}}{p_{i,t-1}}\right) = \sum_{\tau=1}^{\tau=t} \ln\left(\frac{p_{i,\tau}}{p_{i,\tau-1}}\right) = \sum_{\tau} \ln(\ell_\tau).$$

Defintion of a "log-change" (price) index
The logarithm of a log-change (price) index P_{0t}^* or DP_{0t}^*, that is $\ln(P_{0t}^*)$, or $\ln(DP_{0t}^*)$ is a function of logarithmic price relatives Dp_i as defined in eq. 4, such as
(7) $\ln(P_{ot}^*) = \sum g_i(Dp_{0t})$ with weights g_i
The notation DP* is chosen in order to indicate a relationship between a traditional index P* and its "log change" counterpart

Figure *2: Log-change indices

Figure *2 Notation (general structure of the formula)

$\ln\left(P_{ot}^{*}\right)=\sum g_i\left(Dp_{0t}\right)$ with weights g_i and Dp_{0t} = $\ln\frac{p_{it}}{p_{i0}}$, L(x,y) = log. mean, v_{i0}, v_{it} = absolute values, w_{i0}, w_{it} = relative values (value shares) = v/Σv =v/V



1) $\overline{w}_i=\left(w_{it}+w_{it}\right)/2$

2) The name was given because this index has some resemblance to the "normal" index of

$$\text{Walsh}\;\; P_{0t}^{W}=P_{0t}^{W1}=\frac{\sum p_{it}\sqrt{q_{i0}q_{it}}}{\sum p_{i0}\sqrt{q_{i0}q_{it}}}=\sum\frac{p_{it}}{p_{i0}}\frac{\sqrt{\left(p_{i0}q_{i0}\right)\left(p_{i0}q_{it}\right)}}{\sum\sqrt{\left(p_{i0}q_{i0}\right)\left(p_{i0}q_{it}\right)}}.$$

3) Note that in general $\sum_i L\left(v_{i0},v_{it}\right)\neq L\left(\sum v_{i0},\sum v_{it}\right)$.

Uniqueness theorem (UT)

UT: The Cobb Douglas index is the **unique** index function that satisfies

1. the circular test (transitivity) **and**
2. the following five fundamental axioms: monotonicity, linear homogeneity, identity and commensurability.

The axioms constitute a *system* of axioms (independent and consistent) that can be satisfied by one and only one specific index function, the function P^{CD}. Lowe's index, as an example violates commensurability.

The following index function is known as Törnqvist - or Törnqvist - Theil index

(6.4.11) $P_{0t}^{T} = \prod_{i=1}^{n}\left(\frac{p_{it}}{p_{i0}}\right)^{\overline{w}_i}$ where $\overline{w}_i$ is the mean of expenditure shares for period 0 and

period t $\overline{w}_i = \frac{1}{2}\left(w_{i0} + w_{it}\right) = \frac{1}{2}\left(\frac{p_0 q_0}{\Sigma p_0 q_0} + \frac{p_t q_t}{\Sigma p_t q_t}\right)$, or alternatively

(6.4.12) $\ln(P_{0t}^{T}) = \frac{1}{2}\left[\Sigma w_{i0} \ln\left(p_{it} / p_{i0}\right) + \Sigma w_{it} \ln\left(p_{it} / p_{i0}\right)\right] = \frac{1}{2}\left[\ln\left(DP_{0t}^{L}\right) + \ln\left(DP_{0t}^{P}\right)\right]$,

or of course equivalently $\log P_{0t}^{T} = \frac{1}{2}\left[\log\left(DP_{0t}^{L}\right) + \log\left(DP_{0t}^{P}\right)\right]$.

(6.4.12a) $P_{0t}^{T} = \sqrt{DP_{0t}^{L} DP_{0t}^{P}}$, PT satisfies time reversibility but not factor reversibility

(6.4.14a) $DP_{0t}^{L} DQ_{0t}^{P} = \prod_{i=1}^{n}\left(\frac{p_{it}}{p_{i0}}\right)^{w_{i0}} \prod_{i=1}^{n}\left(\frac{q_{it}}{q_{i0}}\right)^{w_{it}} \neq V_{0t}$,

Remark to fig. * 3 (next page)

Vartia (1987) has proved some approximations using covariances and variances that the difference

(6.4.16a) $\log\left(DP_{0t}^{P}\right)$ - $\log\left(DP_{0t}^{L}\right)$ is approximately equal to $\operatorname{cov}\left(\dot{p}, \dot{v}\right)$,

and correspondingly

(6.4.16b) $\log\left(P_{0t}^{P}\right)$ - $\log\left(P_{0t}^{L}\right) \approx \operatorname{cov}\left(\dot{p}, \dot{q}\right)$.

These are covariances between logarithmic deviations $\dot{p}_i, \dot{q}_i$ and $\dot{v}_i$.

(6.5.1) $L(x,y) = L(y,x) = \frac{y - x}{\ln(y / x)}$ if $x \neq y$ and $L(x,y) = x$ if $x = y$.

Figure *.3: A system of six index formulas (Vartia) and the five tined fork of I. Fisher

type of mean	type of weight structure	
	w_{i0}	w_{it}
arithmetic	P^{L} (Laspeyres)	P^{PA} (Palgrave)
geometric	DP^{L} (log. Lasp.)	DP^{P} (log. Paasche)
harmonic	P^{HL} (harm.Lasp)	P^{P} (Paasche)

The five tined fork	
tine	formula
uppermost (+2)	P^{PA}
mid-upper (+1)	DP^{P}
middle (0) unbiased	P^{L}, P^{T} P^{F} , P^{P}
mid lower (-1)	DP^{L}
lower most (-2)	P^{HL}

base weight inequality
$P^{L}_{0t} \geq DP^{L}_{0t} \geq P^{HL}_{0t}$

current weight inequality
$P^{PA}_{0t} \geq DP^{P}_{0t} \geq P^{P}_{0t}$

Fisher's result was
$P^{PA} > DP^{P} > P^{L} > P^{T} > P^{F} > P^{P} > DP^{L} > P^{HL}$

(3) $\sqrt{P^{HL}_{0t} P^{PA}_{0t}}$

(2) Törnqvist $P^{T}_{0t} = \sqrt{DP^{L}_{0t} DP^{P}_{0t}}$

(1) Fisher $P^{F}_{0t} = \sqrt{P^{L}_{0t} P^{P}_{0t}}$

There was hardly any attention given to index no. 3. All three indices (1), (2) and (3) are "unbiased" index formulas and their results are in general in close agreement with one another.

Figure *4: Aggregation of rates of change (growth rates) over commodities (additive model)

Notation (consider change from 0 to t)
individual commodity y_i (i = 1, ... ,n), aggregate $Y_\tau = \sum_i y_{i\tau}$, τ = 0, 1 (additive model)

conventional growth rate	growth rate on the basis of log changes
(6.5.2) $r(Y) = \frac{Y_t - Y_0}{Y_0} = \sum_{i=1}^{n} a_{i0}\left(\frac{y_{it} - y_{i0}}{y_{i0}}\right) = \sum a_{i0} r(y_i)$	(6.5.3) $\ln\left(\frac{Y_t}{Y_0}\right) = \sum_{i=1}^{n} \frac{b_i}{\sum b_i} \ln\left(\frac{y_{it}}{y_{i0}}\right) = \sum \beta_i \ln\left(y_{it}/y_{i0}\right)$
weights $a_{i0} = \frac{y_{i0}}{\sum_i y_{i0}} = \frac{y_{i0}}{Y_0}$	weights $b_i = L\left(a_{i0}, a_{it}\right)$, $a_{it} = \frac{y_{it}}{\sum_i y_{it}}$

commentary on the type of weights

weights a_{i0} make use of the structural information related to period 0 only	b_i being an average of a_{i0} and a_{it} are taking both periods into account*

* weights are "balanced", i.e. they employ structural data of both periods.

(6.5.4) $$\ln\left(\frac{V_t}{V_0}\right) = \sum \ln\left(\frac{v_{it}}{v_{i0}}\right)\frac{L(v_{it}, v_{i0})}{L(V_t, V_0)}$$

(6.5.5) $$L(w_{it}, w_{i0})\ln(v_{it}/v_{i0}) = (w_{it} - w_{i0}) + L(w_{it}, w_{i0})\ln(V_t/V_0)$$

(6.5.6) $$\ln\left(\frac{V_t}{V_0}\right) = \sum\left(\ln\left(\frac{v_{it}}{v_{i0}}\right)\frac{L(w_{it}, w_{i0})}{\sum L(w_{it}, w_{i0})}\right),$$

(6.5.7) $$\textstyle\sum_i L(v_{it}, v_{i0}) \neq L(V_t, V_0) = L(\sum_i v_{it}, \sum_i v_{i0}),$$

(6.5.8) $$\ln\left(P_{0t}^{V1}\right) = \frac{\sum_{i=1}^{n} L(v_{it}, v_{i0})\ln\left(\frac{p_{it}}{p_{i0}}\right)}{L(V_t, V_0)}$$, and correspondingly the quantity index

(6.5.8a) $$\ln\left(Q_{0t}^{V1}\right) = \frac{\sum_{i=1}^{n} L(v_{it}, v_{i0})\ln\left(\frac{q_{it}}{q_{i0}}\right)}{L(V_t, V_0)},$$ (6.5.8b) $$\frac{L(v_{it}, v_{i0})}{L(V_t, V_0)}\ln\left(\frac{q_{it}}{q_{i0}}\right) = \Delta\left(\ln\left(Q_{0t}^{V1}\right)\right)$$

(6.5.9) $$\ln V_{0t} = \Sigma\, \hat{w}_i \ln(v_{it} / v_{i0}),$$

(6.5.10) $$\ln\frac{w_t}{w_0} = \ln\frac{v_t}{v_0} - \ln\frac{\Sigma v_t}{\Sigma v_0} = \ln\frac{v_t}{v_0} - \ln\frac{V_t}{V_0}$$

(6.5.6a) $$\sum L(w_t, w_0)\ln\left(\frac{v_t}{v_0}\right) = \ln\left(\frac{V_t}{V_0}\right)\sum L(w_t, w_0).$$

(6.5.11) $$\ln\left(P_{0t}^{V2}\right) = \sum\frac{L(w_{it}, w_{i0})\ln\left(\frac{p_{it}}{p_{i0}}\right)}{\sum L(w_{it}, w_{i0})} = \sum \hat{w}_{it} \ln(p_{it}/p_{i0})$$

(6.5.11a) $$\ln\left(Q_{0t}^{V2}\right) = \sum \hat{w}_{it} \ln(q_{it}/q_{i0})$$

Figure 6.5.2: Törnqvist, Vartia-II and Walsh-II index

general structure: $P(g) = \Pi\left(\frac{p_{it}}{p_{i0}}\right)^g$

with weights g ($\Sigma g = 1$)

Törnqvist	Vartia II	Walsh II
$g = \bar{w} = \frac{1}{2}(w_0 + w_t)$	$g = \hat{w} = L(w_0, w_t) / \sum L(w_0, w_t)$	$g = \sqrt{v_0 v_t} / \Sigma\sqrt{v_0 v_t}$

$$\sqrt{w_0 w_t} \leq L(w_0, w_t) \leq \tfrac{1}{2}(w_0 + w_t) = \bar{w},\quad \sqrt{w_0 w_t} = \sqrt{v_0 v_t}\Big/\sqrt{V_0 V_t}$$

(6.5.13) $$P_{0t}^{V1}(\mathbf{p}_0, \mathbf{q}_0, \lambda\mathbf{p}_t, \mathbf{q}_t) \neq \lambda P_{0t}^{V1}(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$$ (no linear homogeneity)

(6.5.13a) $P_{0t}^{V1}\left(\mathbf{p}_0,\mathbf{q}_0,\lambda\mathbf{p}_0,\mathbf{q}_t\right) \neq \lambda$ (strict proportionality violated).

$$\ln\left(P_{0t}^{V1}\left(\mathbf{p}_0,\mathbf{q}_0,\lambda\mathbf{p}_0,\mathbf{q}_t\right)\right) = \ln(\lambda)\frac{\sum L(v_{i0},v_{it})}{L(V_0,V_t)}; \qquad L\left(\lambda v_{it}, v_{i0}\right) = \frac{\lambda v_{it} - v_{i0}}{\ln\left(\lambda v_{it}/v_{i0}\right)} \neq \frac{v_{it} - v_{i0}}{\ln\left(v_{it}/v_{i0}\right)}$$

(6.5.14) $\ln\left(P_{0t}^{V2}(\lambda)\right) = \ln(\lambda) + \ln\left(P_{0t}^{V2}\right)$, or (6.5.14) $P_{0t}^{V2}(\lambda) = \lambda P_{0t}^{V2}$

Ideal Indices(satisfying the factor reversal test)

a) Multiplicative approach

(6.6.3) $$\frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_0'\mathbf{q}_0} = \left(\frac{\mathbf{p}_t'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_0}\right)\left(\frac{\mathbf{q}_t'\mathbf{p}_0}{\mathbf{q}_0'\mathbf{p}_0}\right)\left(\frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_t'\mathbf{q}_0}\frac{\mathbf{p}_0'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_t}\right) = (PC^*)(QC^*)(SC^*).$$

b) Fisher's solution

price component PC*	structural change component SC*	quantity component QC*
$\frac{\mathbf{p}_t'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_0} = P_{0t}^{L}$	$\frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_t'\mathbf{q}_0}\frac{\mathbf{p}_0'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_t} = 1 - \frac{C}{V_{0t}}$	$\frac{\mathbf{q}_t'\mathbf{p}_0}{\mathbf{q}_0'\mathbf{p}_0} = Q_{0t}^{L}$

$\sqrt{\frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_t'\mathbf{q}_0}\frac{\mathbf{p}_0'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_t}}$ → new price component

$\sqrt{\frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_t'\mathbf{q}_0}\frac{\mathbf{p}_0'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_t}}$ → new quantity component

new price component $P^F = PC^*(SC^*)^{1/2}$	new quantity component $Q^F = QC^*(SC^*)^{1/2}$
$P_{0t}^{F} = \sqrt{P_{0t}^{L}P_{0t}^{P}} = \sqrt{\frac{\mathbf{p}_t'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_0}\frac{\mathbf{p}_t'\mathbf{q}_t}{\mathbf{p}_0'\mathbf{q}_t}}$	$Q_{0t}^{F} = \sqrt{Q_{0t}^{L}Q_{0t}^{P}} = \sqrt{\frac{\mathbf{q}_t'\mathbf{p}_0}{\mathbf{q}_0'\mathbf{p}_0}\frac{\mathbf{q}_t'\mathbf{p}_t}{\mathbf{q}_0'\mathbf{p}_t}}$

c) Vartias solution **Figure 6.6.4:** Weights in aggregating log-changes,

Single commodity (i) equation (definition): $\ln(v_{it}/v_{i0}) = \ln(p_{it}/p_{i0}) + \ln(q_{it}/q_{i0})$
Aggregation: $\sum g_i \ln(v_{it}/v_{i0}) = \sum g_i \ln(p_{it}/p_{i0}) + \sum g_i \ln(q_{it}/q_{i0})$

Vartia I weights	Vartia II weights
$g_i = \frac{L(v_{i0},v_{it})}{L(V_0,V_t)}$, $V_0 = \sum v_{i0}$, $V_t = \sum v_{it}$.	$g_i = \frac{L(w_{it},w_{i0})}{\sum L(w_{it},w_{i0})}$
$\sum g_i \ln(v_{it}/v_{i0}) = \ln(V_{0t})\sum\left(\frac{v_{it}-v_{i0}}{V_t - V_0}\right)$ $= \ln(V_{0t})$, since $\sum\left(\frac{v_{it}-v_{i0}}{V_t - V_0}\right) = 1$	$\sum g_i \ln(v_{it}/v_{i0}) = \frac{\sum\left(\frac{w_{it}-w_{i0}}{\ln w_{it} - \ln w_{i0}} \ln(v_{it}/v_{i0})\right)}{\sum L(w_{it}/w_{i0})} = \ln(V_{0t})$

* The problem is not to show that $\sum g_i \ln(p_{it}/p_{i0})$ is $\ln\left(P_{0t}^{V1}\right)$, or $\ln\left(P_{0t}^{V2}\right)$ because these indices are defined in this way. The same is true for quantity indices. It only remains to be shown that the weighted aggregation $\sum g_i \ln(v_{it}/v_{i0})$ results in $\ln(V_{0t})$, with the value index V_{0t}.